Math 20D - Fall 2011 - Final Exam Topics

(A.) Modelling with differential equations:
   (i) Mixture problems: \(dQ/dt = \text{rate in} - \text{rate out}\).

(B.) First order equations
   (i) Linear equations:
       \[ y' + p(t)y = q(t). \]
       * Solve by integrating factors. Bring the equation in standard linear form;
       * Integrating factor
       \[ u(t) = \exp\left( \int p(t)\,dt \right) \]
       * Multiply by \(u\), rewrite the equation as
       \[ (uy)' = uq. \]
   (ii) Nonlinear equations
       * Separable
       \[ \frac{dy}{dx} = f(x)g(y). \]
       Separate variables, then integrate.
       * Autonomous equations
       \[ \frac{dy}{dx} = f(y). \]
       Equilibrium solutions are the roots of \(f\).
       * Exact
       \[ M(x, y) + N(x, y)y' = 0. \]
       Check exactness:
       \[ M_y = N_x. \]
       Find a function \(f\) such that \(f_x = M, f_y = N\). Set \(f = \text{constant}\).

(C1.) Second order homogeneous equations
   \[ y'' + p(t)y' + q(t)y = 0. \]
   (i) General facts:
       * superposition: if \(y_1, y_2\) are solutions, \(c_1y_1 + c_2y_2\) is also a solution.
       * fundamental pair of solutions: the Wronskian
       \[ W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{vmatrix} \neq 0. \]
       For a fundamental pair, the general solution is
       \[ y = c_1y_1 + c_2y_2. \]
       * Abel’s theorem
       \[ W(y_1, y_2) = C \exp\left( -\int p(t)\,dt \right). \]
   (ii) Constant coefficient equations: \(p(t) = p, q(t) = q\).
       * Characteristic equation \(r^2 + pr + q = 0\).
* If the roots $r_1, r_2$ are real, then the fundamental solutions are
  
  \[ y_1 = e^{r_1t}, \quad y_2 = e^{r_2t}. \]

* Complex roots: fundamental solutions are the real and imaginary part of \( e^{r_1t} \). If \( r_1 = \alpha + i\beta \), then
  
  \[ y_1 = e^{\alpha t} \cos \beta t, \quad y_2 = e^{\alpha t} \sin \beta t. \]

* Repeated roots \( r_1 = r_2 = \alpha \), fundamental solutions
  
  \[ y_1 = e^{\alpha t}, \quad y_2 = te^{\alpha t}. \]

(C2.) Inhomogeneous second order equations.

(i) General solution

\[ y = y_p + y_h, \]

where \( y_p \) is the particular solution, \( y_h \) is the homogeneous solution.

(ii) Find a particular solution by undetermined coefficients

\[ y'' + py' + qy = g(t). \]

Three cases: \( g(t) \) can be exponential, trigonometric, polynomial.

* For \( g(t) \) polynomial, look for \( y_p \) as a polynomial with undetermined coefficients. Try to guess its degree first.

* For trigonometric \( g(t) \), look for \( y_p = A \cos t + B \sin t \).

* For exponential case \( g(t) = e^{at} \), use

\[ y_p = \frac{e^{at}}{f(a)} \]

with \( f(a) = a^2 + pa + q \). If \( f(a) = 0 \), look for \( y_p = e^{at}(A + Bt) \) for undetermined \( A, B \).

* For a term \( g(t) = e^{at} \times \) polynomial or trigonometric function, substitute \( y = e^{at}u \), find the differential equation for \( u \), then solve for \( u \) by undetermined coefficients.

(iii) Alternatively, you may use variation of parameters

\[ y = u_1(t)y_1(t) + u_2(t)y_2(t) \]

where

\[ u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)} \, dt, \quad u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)} \, dt. \]

(D.) First order systems of equations \( \mathbf{x}' = A\mathbf{x} \).

(i) Find eigenvalues \( \lambda \) of \( A \):

\[ \det (A - \lambda I) = 0 \]

or alternatively for \( 2 \times 2 \) matrices

\[ \lambda^2 - \lambda \text{Tr} A + \det A = 0. \]

Eigenvectors are found by solving the system

\[ (A - \lambda I)\mathbf{v} = \mathbf{0}. \]

(ii) Finding solutions: find eigenvalues \( \lambda_1, \lambda_2 \) with eigenvectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \). General solution

\[ \mathbf{x} = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2. \]

(iii) Repeated eigenvalues \( \lambda_1 = \lambda_2 = \lambda \).
**Defective case:** one eigenvector \( v \). Solutions
\[
\begin{align*}
    x_1 &= e^{\lambda t} v, \\
    x_2 &= e^{\lambda t} (\alpha + tv)
\end{align*}
\]
where \( \alpha \) is a generalized eigenvector
\[
(A - \lambda I) \alpha = v.
\]

(iv) **Phase portraits.**
- saddles: real eigenvalues of opposite sign,
- nodes (sink or source): real distinct eigenvalues of same sign,
- spiral (sink or source) complex eigenvalues. To find the direction of spirals compute the velocity vector at a point on the trajectory.
- improper nodes: repeated eigenvalues.

(v) **Fundamental matrix**
\[
\Psi(t) = [x_1 \ x_2].
\]
The Wronskian is the determinant of the fundamental matrix
\[
W(x_1, x_2) = \det \Psi(t).
\]
The normalized fundamental matrix/matrix exponential:
\[
\Phi(t) = \Psi(t)\Psi(0)^{-1} = e^{At}.
\]
Solution to IVP \( x' = Ax \) and \( x(0) = x_0 \) is
\[
x = e^{At}x_0.
\]

(vi) **Undetermined coefficients for systems.**

(vii) **Variation of parameters** for systems
\[
x' = Ax + g(t).
\]
A particular solution is given by
\[
x_p = \Psi(t) \int \Psi(t)^{-1} g(t) \, dt.
\]
General solution is
\[
x = x_p + x_h
\]
where \( x_h \) solves the homogeneous system.

(E.) **Series solutions.**
- (i) Radius of convergence.
- (ii) Finding the recurrence relations between the coefficients.

(F.) **Laplace transform.**
- (i) When \( f \) has exponential growth,
\[
f \sim F(s) = \int_0^\infty e^{-st} f(t) \, dt.
\]
- (ii) Laplace transforms of the standard functions:
\[
1 \sim \frac{1}{s}, \quad t^n \sim \frac{n!}{s^{n+1}}, \quad e^{at} \sim \frac{1}{s-a}, \quad e^{at} f(t) \sim F(s-a)
\]
\[
\sin(at) \sim \frac{a}{s^2 + a^2}, \quad \cos(at) \sim \frac{s}{s^2 + a^2}.
\]

(iii) Solving differential equations with Laplace transform:

\[
f' \sim sF(s) - f(0),
\]

\[
f'' \sim s^2 F(s) - sf(0) - f'(0).
\]

(iv) Discontinuous functions:

\[
u_a(t) \sim \frac{e^{-sa}}{s},
\]

\[
u_a(t)f(t - a) \sim e^{-sa} F(s).
\]