Math 220, Problem Set 2. Due Wednesday, October 21.

1. Calculate the following integrals:

(i) \[ \int_{|z|=2} \frac{e^z}{(z-1)(z-3)^2} \, dz \]

(ii) \[ \int_{|z|=2} \frac{\sin z}{z+i} \, dz \]

(iii) \[ \int_{|z|=1} \frac{e^z}{(z-2)^3} \, dz \]

(iv) \[ \int_{|z+2i|=a} \frac{dz}{z^2 + 1} \]

(v) \[ \int_{|z|=1} \frac{dz}{z^5 + iz - 4} \]

2. Let \( p \) be a complex polynomial. Assume that \( f : \mathbb{C} \to \mathbb{C} \) is a holomorphic function such that
\[ |f(z)| \leq |p(z)|, \]
for \( |z| \) sufficiently large. Show that \( f \) must be a polynomial.

3. Let \( f : \mathbb{C} \to \mathbb{C} \) be a holomorphic function.

(i) If \( \text{Re} \, f \) is bounded (either from below or from above), show that \( f \) is constant. Hint: \( e^{\pm f(z)} \).

(ii) Assume now that
\[ \text{Re} \, f(z) \leq \text{Im} \, f(z) \]
for all \( z \in \mathbb{C} \). Show that \( f \) is constant. Hint: \( e^{(1+i)f(z)} \).

4. The Bernoulli numbers are defined via the power series expansion
\[ \frac{z}{e^z - 1} = \sum_{j=0}^{\infty} B_j \frac{z^j}{j!} \]

(i) Calculate the first three non-zero Bernoulli numbers.

(ii) Show that \( B_{2k+1} = 0 \) for \( k \geq 1 \).

(iii) The formulas
\[ 1 + 2 + \ldots + N = \frac{N(N+1)}{2} \]
\[ 1^2 + 2^2 + \ldots + N^2 = \frac{N(N+1)(2N+1)}{6} \]
are well-known. Prove that

\[ 1^p + 2^p + \ldots + N^p = \frac{1}{p+1} \sum_{j=0}^{p} (-1)^j B_j \binom{p+1}{j} N^{p+1-j}. \]

What does this formula give for \( p = 1, 2, 3 \)?

5. Find all entire functions that satisfy

\[ |f(z)|^2 \leq |z|. \]

6. Consider the holomorphic function

\[ f(z) = e^z + i e^{-z} \]

defined over the closed rectangle with corners \( \pm 1 \pm \frac{i\pi}{2} \). Find the points where the maximum of \( |f| \) occurs and confirm that they lie on the boundary. Where does the minimum occur?

7. Show that an entire function \( f : \mathbb{C} \to \mathbb{C} \) which is doubly periodic must be constant. A doubly periodic function has the property that

\[ f(z) = f(z + \omega_1) = f(z + \omega_2) \]

for two complex numbers \( \omega_1, \omega_2 \) such that \( \omega_1/\omega_2 \not\in \mathbb{R} \).