Consider the function \( f(z) = ze^{3z} - 1 \). How many zeros does \( f \) have inside the disc \( \Delta(0, 1) \)?

**Answer:** We apply Rouché’s theorem to \( f(z) = ze^{3z} - 1 \), \( g(z) = ze^{3z} \) over the curve \( |z| = 1 \). We have

\[
|f - g| = 1, \quad |g| = |ze^{3z}| = |z| \cdot e^{3-Re z} = e^{3-Re z} \geq e^2 > |f - g|
\]

where we used \( Re z \leq 1 \) for \( |z| = 1 \). Therefore, \( f \) and \( g \) have the same number of zeroes inside the unit disc. Clearly \( g \) vanishes only at \( z = 0 \) with order 1, hence \( f \) must have only one simple zero in \( \Delta(0, 1) \) as well.
Problem 2.

Consider a meromorphic function $g$ over $\mathbb{C}$ with simple poles with integer residues at the poles. Show that there exists a meromorphic function $f$ on $\mathbb{C}$ such that $f'/f = g$.

Answer: Let $a_n$ be the poles of $g$ and let $m_n \in \mathbb{Z}$ be the residues. Clearly either $\{a_n\}$ is a finite sequence or $a_n \to \infty$, since otherwise $\{a_n\}$ will have an accumulation point in $\mathbb{C}$. Write:

- $a^1_n$ for the subsequence corresponding the residues $m^1_n \geq 0$,
- $a^2_n$ for the subsequence with residues $m^2_n < 0$.

We consider the Weierstrass entire functions

- $f_1$ with zeros at $a^1_n$ of order $m^1_n$;
- $f_2$ with zeros at $a^2_n$ of order $-m^2_n$.

These can be constructed as $a^j_n$ are either finite sequences or their absolute values converge to $\infty$.

Let $F = f_1/f_2$. Then $F$ is meromorphic over $\mathbb{C}$ and it has zeros at $a^1_n$ and poles at $a^2_n$, thus the zeros and poles of $F$ are at $a_n$ and the order of each zero/pole is $m_n$. By the proof of the argument principle, $F'/F$ has simple poles at $a_n$ and

$$\text{Res} \left( \frac{F'}{F}, a_n \right) = \text{Ord}(F, a_n) = m_n.$$  

Thus, $F'/F - g$ has simple poles at $a_n$ and the residue equals 0 at $a_n$. Thus

$$F'/F - g = h$$

can be extended to an entire function. Let $H$ be a primitive of $h$, and set

$$f = Fe^{-H}.$$  

Then, taking the logarithmic derivative,

$$f'/f = F'/F - H' = F'/F - h = g.$$  

This completes the proof.
Problem 3.

Recall the function

\[ G(z) = \prod_{n=1}^{\infty} E_1 \left( -\frac{z}{n} \right). \]

(i) Examining the location of zeros, show that

\[ \left( z + \frac{1}{2} \right) G(z) G \left( z + \frac{1}{2} \right) = e^{h(z)} G(2z), \]

for some entire function \( h \). Show furthermore that \( h(z) = az + b \).

(ii) Deduce from (i) that

\[ \Gamma(z) \Gamma \left( z + \frac{1}{2} \right) = e^{az+b} \Gamma(2z), \]

for some new constants \( a, b \).

(iii) Derive Legendre’s duplication formula

\[ 2^{2z-1} \Gamma(z) \Gamma \left( z + \frac{1}{2} \right) = \sqrt{\pi} \Gamma(2z). \]

Answer:

(i) *The function \( G(z) \) has zeros at all negative integers \( \{ -1, -2, \ldots \} \). The function \( G \left( z + \frac{1}{2} \right) \) has zeros at \( \{ -\frac{3}{2}, -\frac{5}{2}, \ldots \} \). Thus

\[
\left( z + \frac{1}{2} \right) G(z) G \left( z + \frac{1}{2} \right) = e^{h(z)} G(2z),
\]

for some entire function \( h(z) \).

Next, we compute the logarithmic derivatives of both sides. This yields

\[
\frac{1}{z + \frac{1}{2}} + \frac{G'(z)}{G(z)} + \frac{G'(z + \frac{1}{2})}{G \left( z + \frac{1}{2} \right)} = h'(z) + 2 \frac{G'(2z)}{G(2z)}.
\]

Since

\[
G(z) = \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}},
\]

we have

\[
\frac{G'(z)}{G(z)} = \sum_{n=1}^{\infty} \left( \frac{1}{z + n} - \frac{1}{n} \right).
\]

Furthermore,

\[
\frac{G' \left( z + \frac{1}{2} \right)}{G \left( z + \frac{1}{2} \right)} = \sum_{n=1}^{\infty} \left( \frac{1}{z + \frac{1}{2} + n} - \frac{1}{n} \right).
\]
Finally,
\[
\frac{G'(2z)}{G(2z)} = \sum_{n=1}^{\infty} \left( \frac{1}{2z + n} - \frac{1}{n} \right).
\]

This yields
\[
h'(z) = \frac{1}{z + \frac{1}{2}} + \sum_{n=1}^{\infty} \left( \frac{1}{z + n} - \frac{1}{n} \right) + \sum_{n=1}^{\infty} \left( \frac{1}{z + \frac{1}{2} + n} - \frac{1}{n} \right) - 2 \cdot \sum_{n=1}^{\infty} \left( \frac{1}{2z + n} - \frac{1}{n} \right).
\]

In the last sum, the terms with \( n \to 2n \) become
\[
2 \cdot \sum_{n=1}^{\infty} \left( \frac{1}{2z + 2n} - \frac{1}{2n} \right) = \sum_{n=1}^{\infty} \left( \frac{1}{z + n} - \frac{1}{n} \right)
\]
so they cancel the first sum. For \( n \to 2n + 1 \), the terms involving \( z \) match with the terms in the second sum, but the free terms are
\[
\frac{-1}{n} + \frac{2}{2n + 1} = \frac{-1}{n(2n + 1)}.
\]

Thus \( h'(z) \) does not depend on \( z \), showing \( h' \) is constant. This gives \( h(z) = az + b \).

(ii) Using
\[
\Gamma(z) = \frac{e^{-\gamma z} - 1}{z G(z)}
\]
we have
\[
\Gamma(z)\Gamma \left( z + \frac{1}{2} \right) = e^{-\gamma z} e^{-\gamma(z+\frac{1}{2})} \frac{1}{z (z + \frac{1}{2})} \Gamma(z) G(z) G \left( z + \frac{1}{2} \right) = 2e^{-\gamma/2} \frac{e^{-2\gamma}}{2z} \frac{1}{e^{az+b} G(2z)} = e^{a'z+b'} \Gamma(2z).
\]

We relabel \( a', b' \) by \( a, b \).

(iii) Setting \( z = \frac{1}{2} \) and using \( \Gamma(1) = 1, \Gamma(1/2) = \sqrt{\pi} \), we obtain
\[
\Gamma \left( \frac{1}{2} \right) \Gamma(1) = e^{a'/2+b'} \Gamma(1) \implies e^{a'/2+b'} = \sqrt{\pi}.
\]

Setting \( z = 1 \) and using \( \Gamma(1) = 1, \Gamma(2) = 1, \Gamma \left( \frac{3}{2} \right) = \frac{1}{2} \Gamma \left( \frac{1}{2} \right) = \frac{\sqrt{\pi}}{2} \) we obtain
\[
e^{a+b} = \frac{\sqrt{\pi}}{2}.
\]

Thus
\[
e^{a/2} = \frac{1}{2}, e^{b} = 2\sqrt{\pi} \implies e^{a+b} = \sqrt{\pi}2^{-2z+1}.
\]

The result follows by substitution.
Problem 4.

Assume that $f$ is entire and $f(z) = f(z + 1)$ such that $|f(z)| \leq e^{|z|}$. Show that $f$ is constant.

(i) Consider

$$g(z) = \frac{f(z) - f(0)}{\sin \pi z}.$$ 

Show that $g$ is periodic and that $g$ can be extended to an entire function.

(ii) By direct calculation, show that $g$ is bounded in the strip $0 \leq \text{Re } z \leq 1$.

(iii) Conclude from (ii) that $g = 0$ hence $f$ is constant.

Answer:

(i) We have

$$g(z + 1) = \frac{f(z + 1) - f(0)}{\sin \pi (z + 1)} = \frac{f(z) - f(0)}{-\sin \pi z} = -g(z).$$

Thus

$$g(z + 1) = -g(z) \implies g(z) = g(z + 2).$$

To show that $g$ is entire, it suffices to show that $g$ has removable singularities at $z = n$, $n \in \mathbb{Z}$. By periodicity $g(z + 1) = -g(z)$, it suffices to prove that $g$ has removable singularity at $z = 0$. But

$$\lim_{z \to 0} g(z) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} \cdot \frac{z}{\sin \pi z} = f'(0) \cdot \frac{1}{\pi}$$

hence $g$ is bounded near 0, so the singularity is removable.

(ii) Write $z = x + iy$ with $0 \leq x \leq 1$. We have

$$|g(z)| \leq \frac{|f(z)| + |f(0)|}{|\sin \pi z|} \leq \frac{e^{|z|} + |f(0)|}{|\sin \pi z|}.$$ 

Since $|z| = |x + iy| \leq |x| + |y| \leq 1 + |y|$, we have $e^{|z|} \leq e^{|y| + 1}$. Assume $y > 0$, the argument for $y < 0$ being similar.

$$|2 \sin \pi z| = |e^{\pi iz} - e^{-\pi iz}| = |e^{\pi ix}e^{-\pi y} - e^{-\pi ix}e^{\pi y}| \geq e^{-\pi y}|e^{\pi x} - e^{-\pi x}| = e^{-\pi y} > 0.$$ 

Thus

$$|g(z)| \leq 2 \frac{e^{y+1} + |f(0)|}{e^{\pi y} - e^{-\pi y}}.$$ 

The last expression converges to 0 as $y \to \infty$, so it is bounded. Thus $|g(z)|$ is bounded in the strip $0 \leq \text{Re } z \leq 1$.

(iii) Since $g$ is bounded for $0 \leq \text{Re } z \leq 1$, it follows by periodicity that $g$ is bounded over the complex plane. By Liouville’s theorem, $g$ is constant. We have also seen that $\lim_{y \to \infty} g(z) = 0$ over the domain in (ii), hence this constant must vanish. Thus $g(z) = 0 \implies f(z) = f(0)$ showing $f$ is constant.
Problem 5.

Consider $f : \Delta(0, 1) \to \mathbb{C}$ holomorphic and nonconstant, and define $M(r) = \max_{|z| = r} \text{Re}(f(z))$ for $0 \leq r < 1$. Show that $M : [0, 1) \to \mathbb{R}$ is strictly increasing.

Answer: The function $g(z) = e^{f(z)}$ is holomorphic over $\Delta(0, 1)$. We let

$$N(r) = \max_{|z| = r} |g(z)|.$$

Since $|e^w| = e^{\text{Re}(w)}$, it follows that $N(r) = e^{M(r)}$. It suffices to show that the function $N$ is strictly increasing.

Let $r_1 < r_2$. Use the maximum principle over the disc $\overline{\Delta}(0, r_2)$. The maximum of $|g|$ over $\overline{\Delta}(0, r_2)$ must be achieved over the boundary, hence

$$N(r_2) = \max_{|z| \leq r_2} |g(z)| \geq N(r_1) = \max_{|z| = r_1} |g(z)|,$$

since the circle $|z| = r_1$ is contained in $\overline{\Delta}(0, r_2)$. If we had equality, then there would be an interior point, namely a point on the circle $|z| = r_1$, which achieves the maximum of $|g|$ over $\overline{\Delta}(0, r_2)$. Therefore, $g$ is constant in $\Delta(0, r_2)$ so that $g(z) = c \neq 0$ over $\Delta(0, r_2)$. This implies

$$f(z) = \log c + 2\pi in_z$$

for some $n_z \in \mathbb{Z}$ that may depend on $z$, and for some choice of logarithm. By continuity of $f$, $n_z$ must be be constant, hence $f = K$ constant in $\Delta(0, r_2)$. The zeros of $f - K$ would then not be isolated in $\Delta(0, 1)$, hence $f - K = 0$ in $\Delta(0, 1)$. This is however not allowed, as $f$ is not constant. Hence equality cannot occur and $N(r_1) < N(r_2)$. The proof is completed.
Problem 6.

Assume that $f : \mathbb{C} \to \mathbb{C}$ is entire and injective. Show that $f(z) = az + b$.

**Answer:** Consider $g(z) = f\left(\frac{1}{z}\right)$ which is holomorphic over $\mathbb{C} \setminus \{0\}$ and also injective. If $0$ is a pole for $g$, we consider the Laurent expansion of $g$:

$$g(z) = \sum_{n=-N}^{\infty} a_n z^n \implies f(z) = \sum_{n=-N}^{\infty} a_n z^{-n}.$$  

Since $f$ is entire, we must have $a_n = 0$ for $n > 0$, hence

$$f(z) = \sum_{n=-N}^{0} a_n z^{-n}.$$  

This means $f$ is a polynomial, and since $f$ is injective it follows that $f$ has degree 1. Otherwise, if $\deg f \geq 2$, then $f(z) - f(a)$ has $z = a$ as its only root by injectivity, hence $z = a$ must be a root of $f - f(a)$ with multiplicity $\deg f \geq 2$. Therefore, the derivative $(f(z) - f(a))'|_{z=a} = 0$ which implies $f'(a) = 0$ for all $a$, hence $f$ is constant. This is a contradiction.

If $z = 0$ is an essential singularity for $g$, then take $r > 0$. By the open mapping theorem $g(\mathbb{C} \setminus \Delta(0, r))$ is an open set. By the Caseroti-Weierstrass theorem, $g(\Delta(0, r) \setminus \{0\})$ is dense, hence it must intersect the open set $g(\mathbb{C} \setminus \Delta(0, r))$. This however contradicts the fact that $g$ is injective.
Problem 7.

Let \( R(z) = \frac{P(z)}{Q(z)} \) be a rational function such that \( \deg P + 2 \leq \deg Q \). Assume that \( Q \) has simple zeros at \( a_1, \ldots, a_q \), where \( a_j \in \mathbb{C} \setminus \mathbb{Z} \). Show that

\[
\sum_{m=-\infty}^{\infty} R(m) = -\pi \sum_{j=1}^{q} \frac{P(a_j)}{Q'(a_j)} \cdot \cot \pi a_j.
\]

(i) Let \( \gamma_n \) be the square with corners

\[
\pm \left( n + \frac{1}{2} \right) \pm i \left( n + \frac{1}{2} \right).
\]

Show that there exist constants \( M_1, M_2 > 0 \) such that if \( n \) is sufficiently large, and \( z \) is on the curve \( \gamma_n \), we have

\[
|\pi \cot \pi z| \leq M_1
\]

and

\[
|R(z)| \leq M_2 |z|^{-2}.
\]

(ii) Show that

\[
\lim_{n \to \infty} \int_{\gamma_n} R(z) \pi \cot \pi z \, dz = 0.
\]

(iii) Using (i) conclude that \( \sum_{m=-\infty}^{\infty} R(m) \) converges.

(iv) Show that \( \pi \cot \pi z \) has poles at all integers \( m \in \mathbb{Z} \) with residue equal to 1. Find the poles and residues of \( R(z) \pi \cot \pi z \). Finally, conclude the argument.

Answer:

(i) We have

\[
\lim_{|z| \to \infty} R(z) z^2 = \lim_{|z| \to \infty} \frac{P(z) z^2}{Q(z)} = \alpha
\]

where \( \alpha \) denotes the quotient of leading terms in \( P \) and \( Q \) if \( \deg P + 2 = \deg Q \) and \( \alpha = 0 \) otherwise. Thus, for \( |z| > \eta \) we have

\[
|R(z) z^2| < \alpha + 1 \implies |R(z)| \leq \frac{\alpha + 1}{|z|^2} = M_2.
\]

To show the claim about the cotangent, it suffices by the fact that cotangent is odd, to consider only two sides of the square, for instance the sides:

\[
y = n + \frac{1}{2}, |x| \leq n + \frac{1}{2} \text{ and } x = n + \frac{1}{2}, |y| \leq n + \frac{1}{2}.
\]

We compute

\[
|\cot \pi z| = \left| \frac{e^{\pi iz} + e^{-\pi iz}}{e^{\pi iz} - e^{-\pi iz}} \right| = \left| \frac{e^{2\pi iz} + 1}{e^{-2\pi iz} - 1} \right| \leq 1 + \frac{2}{|e^{-2\pi iz} - 1|}.
\]
We will show \( |e^{-2\pi i z} - 1| > 1 \) over the two sides, completing the argument. Indeed, over the side \( y = n + \frac{1}{2} \), we have

\[
|e^{-2\pi i z} - 1| = |e^{-2\pi i x} e^{2\pi (n+1/2)} - 1| \geq e^{2\pi (n+1/2)} - 1 > 1.
\]

Over the side \( x = n + \frac{1}{2} \), we have

\[
|e^{-2\pi i z} - 1| = |e^{-2\pi i(n+1/2) + i2\pi y} - 1| = |e^{2\pi y} - 1| = e^{2\pi y} + 1 > 1.
\]

(ii) Using (i) we have

\[
|R(z)\pi \cot \pi z| \leq \frac{M_1 M_2}{|z|^2} \leq \frac{M_1 M_2}{(n + 1/2)^2}.
\]

Thus

\[
\left| \int_{\gamma_n} R(z)\pi \cot \pi z \, dz \right| \leq \frac{M_1 M_2}{(n + 1/2)^2} \cdot \text{length} (\gamma_n) = \frac{M_1 M_2}{(n + 1/2)^2} \cdot 4(2n + 1) \to 0
\]
as \( n \to \infty \).

(iii) We have \( |R(m)| \leq \frac{M_2}{m^2} \) for \( m \) large, so \( \sum_{m=-\infty}^{\infty} R(m) \) converges by the comparison test, since \( \sum_{m=-\infty, \text{m} \neq 0}^{\infty} \frac{1}{m^2} \) converges.

(iv) Clearly \( \cot \pi z = \frac{\cos \pi z}{\sin \pi z} \) has poles whenever \( \sin \pi z = 0 \) so for \( z = m, m \in \mathbb{Z} \). By the rules of computing residues, we have

\[
\text{Res}_{z=m}(\pi \cot \pi z) = \text{Res}_{z=m} \left( \frac{\pi \cos \pi z}{\sin \pi z} \right) = \frac{\pi \cos \pi z}{(\sin \pi z)'} \bigg|_{z=m} = \frac{\pi \cos \pi z}{\pi \cos \pi z} \bigg|_{z=m} = 1.
\]

The function \( R(z)\pi \cot \pi z \) has poles at \( z = m \) and at \( z = a_j \). Clearly, since \( R \) is holomorphic near \( m \), we have

\[
\text{Res}_{z=m}(\pi \cot \pi z R(z)) = \text{Res}_{z=m} \left( \frac{1}{z-m} + \ldots \right) (R(m) + (z-m)R'(m) + \ldots) = R(m).
\]

Similarly, since \( \cot \pi z \) is holomorphic near \( a_j \), and \( R \) has a simple pole at \( a_j \) with residue \( \frac{P(a_j)}{Q'(a_j)} \), we have

\[
\text{Res}_{z=a_j}(\pi \cot \pi z R(z)) = \pi \cot \pi a_j \cdot \frac{P(a_j)}{Q'(a_j)}.
\]

Putting everything together via the residue theorem,

\[
0 = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma_n} R(z)\pi \cot \pi z \, dz = \sum_{m=-\infty}^{\infty} R(m) + \sum_{j=1}^{q} \pi \cot \pi a_j \cdot \frac{P(a_j)}{Q'(a_j)}.
\]

This is what we set out to prove.
Problem 8.

(i) Write down an entire function with only simple zeros at \( z = \sqrt{n} \) for each \( n \in \mathbb{Z}_{\geq 0} \).

(ii) Write down a meromorphic function on \( \mathbb{C} \) only with simple poles at \( z = n \) and residues equal to \( \sqrt{n} \) for each \( n \in \mathbb{Z}_{\geq 0} \).

Answer:

(i) The function

\[
 f(z) = z \prod_{n=1}^{\infty} E_2 \left( \frac{z}{\sqrt{n}} \right)
\]

is an example. This works since

\[
 \sum_{n=1}^{\infty} \frac{1}{(\sqrt{n})^{2+1}} < \infty.
\]

(ii) The problem is asking for the Mittag-Leffler solution associated to the poles \( z = n \) and principal parts \( \frac{\sqrt{n}}{z-n} \), for \( n > 0 \) integer. We determine the convergence enhancing factors by looking at the Laurent expansion

\[
 \frac{\sqrt{n}}{z-n} = -\frac{\sqrt{n}}{n} \cdot \frac{1}{1 - \frac{z}{n}} = -\frac{1}{\sqrt{n}} \left( \frac{1}{n} + \frac{1}{n^2} + \ldots \right).
\]

We have

\[
 \left| \frac{\sqrt{n}}{z-n} + \frac{1}{\sqrt{n}} \right| = |z| \cdot \frac{1}{\sqrt{n}|z-n|}.
\]

We let \( r_n = n^{\frac{1}{4}} \), and estimate for \( |z| < r_n \) that

\[
 \left| \frac{\sqrt{n}}{z-n} + \frac{1}{\sqrt{n}} \right| = |z| \cdot \frac{1}{\sqrt{n}|z-n|} \leq r_n \cdot \frac{1}{\sqrt{n}(n-r_n)} := c_n.
\]

Note that

\[
 \sum_{n=1}^{\infty} c_n < \infty
\]

by the limit comparison test. Indeed,

\[
 \lim_{n \to \infty} \frac{c_n}{\frac{1}{n^2}} = 1
\]

and \( \sum_{n=1}^{\infty} \frac{1}{n^{\frac{7}{4}}} < \infty \). By the proof of Mittag-Leffler, the series

\[
 g(z) = \sum_{n=1}^{\infty} \left( \frac{\sqrt{n}}{z-n} + \frac{1}{\sqrt{n}} \right)
\]

defines a meromorphic function with poles at \( z = n \) and residues \( \sqrt{n} \).