Math 220, Practice problems for the midterm.

1. Let \( f : U \to \mathbb{C} \) be a holomorphic function defined in a connected open set \( U \). Assume that for each \( z \in U \), there are positive integers \( m \) and \( n \) (that may depend on \( z \)) such that
\[
f(z)^m = f(z)^n.
\]
Show that \( f \) is a constant.

*Hint: Open mapping theorem. What does it mean that \( w^m = \overline{w}^n \)?*

2. Let \( f : U \to \mathbb{C} \) be a holomorphic function, let \( a \in U \), and assume that \( f(a) = 0, f'(a) \neq 0 \). Show that if \( r \) is sufficiently small then
\[
\int_{|z-a|=r} \frac{dz}{f(z)} = \frac{2\pi i}{f'(a)}.
\]

3. Find the Laurent expansions around \( 0 \) for the function
\[
f(z) = \frac{z}{(z+1)(z+4)}
\]
valid in three different regions of the complex plane.

4. Calculate the following integrals:
   (i)
   \[
   \int_{|z-1|=1} \frac{\sin(\pi z)}{(z^2 - 1)^2} \, dz
   \]
   (ii)
   \[
   \int_{|z-1|=a} \frac{e^z}{z^2 - 2z} \, dz.
   \]

5. Show that there is no meromorphic function \( f \) on the unit disc \( \Delta(0,1) \) such that \( f' \) has a simple pole at \( z = 0 \).

6. Let \( \Delta \) be the open unit disc. Let \( f : \overline{\Delta} \to \mathbb{C} \) be a continuous function on the closed unit disc, holomorphic on the open disc \( \Delta \). Assume that \( f(\partial \Delta) \subset \partial \Delta \) to the boundary of \( \Delta \). Show that \( f(\Delta) \subset \Delta \).

7. (i) Find the residue at \( z = -1 - i \) for the function
\[
f(z) = \frac{z\Log(z)}{(z + 1 + i)^2}.
\]
Here, the principal branch of the logarithm is used.

(ii) For what value of \( a \), does the function
\[
\frac{1}{e^z - 1} + \frac{a}{\sin z}
\]
has a removable singularity at the origin?
8. Assume that $f : \Delta \to \mathbb{C}$ is a continuous function defined on a closed disc $|z| \leq r$ and holomorphic inside the disc $|z| < r$. Prove that Cauchy’s integral formula

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta - z} d\zeta$$

holds for $|z| < r$.

*Hint: This requires an argument; the usual Cauchy integral formula stated in class does not apply directly. Instead use circles of radii $r - \frac{1}{n}$ and let $n \to \infty$.*

9. Show that there are no bounded nonconstant holomorphic functions

$$f : \mathbb{C} \setminus \{0\} \to \mathbb{C}.$$ 

10. Calculate the following integrals using the methods developed in this course. Carefully explain any estimates you might use.

(i) $$\int_0^\infty \frac{dx}{x^6 + 1}$$

(ii) $$\int_{|z|=r} \frac{dz}{(z-a)^m(z-b)^n}, a, b \in \mathbb{C}, n, m \in \mathbb{Z}_{\geq 0}$$

(iii) $$\int_0^{\pi/2} \frac{dx}{a + \sin^2 x}$$

(iv) $$\int_0^\infty \frac{x \sin x}{(x^2 + a^2)^2} dx, a \in \mathbb{R}.$$ 

11. Re-prove the Casorati-Weierstrass theorem: if $f : \Delta \setminus \{0\} \to \mathbb{C}$ is a holomorphic function on the punctured unit disc with an essential singularity at the origin, then $f(\Delta(0,1) \setminus \{0\})$ is dense in $\mathbb{C}$.

12. Show that $\sin z$ and $\cos z$ are entire functions, and therefore, by Liouville, they cannot be bounded. Explicitly calculate

$$|\sin z|$$

in terms of the real and imaginary part of $z = x + iy$, and explain that in fact $\sin z$ and $\cos z$ are unbounded on $\mathbb{C}$. Nonetheless, show that

$$\sin^2 z + \cos^2 z = 1.$$ 

13. Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Let $\gamma_n$ be the boundary of the rectangle with corners $n + \frac{1}{2} + ni, -n - \frac{1}{2} + ni, -n - \frac{1}{2} - ni, n + \frac{1}{2} - ni$. Evaluate

$$\int_{\gamma} \frac{\pi \cot \pi z}{z^2 - a^2} \, dz$$
via the residue theorem, and use this to show that

\[ \pi \cot \pi a = \frac{1}{a} + 2a \sum_{n=1}^{\infty} \frac{1}{a^2 - n^2}. \]

*Hint: For \( z = x + iy \), show that \(|\cot \pi z| \leq 2\) on \( \gamma_n \) for \( n \) large. You will need to use the previous problem.*