Math 220A - Fall 2015 - Midterm

Name: ________________________________

Student ID: __________________________

Instructions:

Please print your name and student ID.

There are 6 questions which are worth 70 points. You have 75 minutes to complete the test.

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Problem 1. [10 points.]

Find the Laurent expansions around $z = 0$ of the meromorphic function

$$f(z) = \frac{z}{z^2 - 4}$$

valid in different regions of the complex plane.
Problem 2. [10 points.]

Let \( a, b \neq 0 \) be real numbers and let \( U \) be a connected open set. Let \( f : U \to \mathbb{C} \) be a holomorphic function. Show that if \( a \Re f + b \Im f \) is constant, then \( f \) is constant.
Problem 3. [20 points; 10, 10.]

Calculate the following integrals. Make sure you explain all the estimates you use in your calculation.

(i) 
\[ \int_0^{\infty} \frac{x^2 \, dx}{(x^2 + 1)(x^2 + 4)}. \]
(ii)

\[ \int_0^\infty \frac{\cos ax}{(x^2 + b^2)^2} \, dx, \quad \text{for } a, b > 0. \]
Problem 4. [10 points.]

Let $\gamma_n$ be the boundary of the rectangle with corners
\[ \pm \left( n + \frac{1}{2} \right) \pm i \left( n + \frac{1}{2} \right) \]

Evaluate the integral
\[ I_n = \int_{\gamma_n} \frac{1}{z^2 \sin \pi z} \, dz. \]

Next, show that $\lim_{n \to \infty} I_n = 0$ and deduce from here the identity
\[ \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} = -\frac{\pi^2}{12}. \]
Problem 5. [10 points.]

Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic function. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$. 
Problem 6. [10 points.]

Let $f$ be a nonconstant continuous function in the closed unit disc $\overline{\Delta}$, holomorphic inside the unit disc $\Delta$. Assume that

$$|f(z)| = 1 \text{ for all } |z| = 1.$$  

Show that $f$ must have at least one zero inside $\Delta$. 