Name: __________________________________________

Student ID: _________________________________

Instructions:

Please print your name and student ID.
During the test, you may not use books, calculators or telephones.
Read each question carefully, and show all your work.
There are 10 questions which are worth 115 points. You have 3 hours to complete the test.

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Problem 1. [10 points.]

The parabolic cylinder

\[ y = 1 - x^2 \]

(aligned along the z-axis) is cut by the plane \( y = 0 \), as well as by the planes \( z = 0 \) and \( z = y \). Find the volume of the solid thus obtained.
**Problem 2.** [10 points.]

Find the volume of the region in the first octant bounded by the hyperbolic cylinders

\[ 1 \leq yz \leq 4, \quad 1 \leq xz \leq 9, \quad 4 \leq xy \leq 9. \]
Problem 3. [10 points; 3, 4, 3.]

Consider the field

$$\mathbf{F} = (axyz^3 + 4xy, bx^2z^3 + cx^2 + 2z^2, 3x^2yz^2 + 4yz + 1).$$

(i) For what values of the constants $a, b, c$, is $\mathbf{F}$ a gradient field?
(ii) Find a potential function for the field $\mathbf{F}$.

(ii) Determine the integral $\int_C W_\mathbf{F}$ where $C \subset \mathbb{R}^3$ is the curve parametrized by

$\gamma : [0, 1] \to \mathbb{R}^3$, $\gamma(t) = (e^t \sin(\pi t^4), t^4 - t e^{t^2} - 1, t^5)$.
Problem 4. [12 points; 6, 6.]

Fix $a, h > 0$. Consider the field

$$\mathbf{F} = \langle x^2, xy, -2xz \rangle.$$

Let $D \subset \mathbb{R}^3$ be the “half solid cylinder” contained between:

- the cylindrical surface $x^2 + y^2 = a^2$ and the $yz$-plane (laterally);
- the plane $z = 0$ on the bottom, and the plane $z = h$ on top.

(i) Find the outward flux of the field over the boundary of $D$. 

(ii) Calculate the divergence of the field $\mathbf{F}$.

Next, verify that the divergence theorem holds true for the field $\mathbf{F}$ and the closed half cylinder above. That is, write down the divergence theorem, compute both sides, and check that they are equal.
Problem 5. [10 points.]

Consider the surface $S \subset \mathbb{R}^4$ given by the equations

$$x^2 - yz + zw = 1, \quad xw^2 + x^3 + zw = 2$$

oriented by the form

$$\Omega = dy \wedge dw$$

in a neighborhood of the point $p = (1, 1, 0, 1)$. Write down a positive basis of $S$ at the point $p = (1, 1, 0, 1)$. 
Problem 6.  [10 points; 6, 4.]

Consider the surface $M \subset \mathbb{R}^3$ given by $x^4 + y^4 + z^4 = 1$ and let

$$\mathbf{F} = \langle x^3 z, y^3 z, z \rangle.$$

(i) Calculate $\nabla \times \mathbf{F}$, and show that the flux of $\nabla \times \mathbf{F}$ over any piece $X$ of the surface $M$ equals

$$\iint_X \Phi_{\nabla \times \mathbf{F}} = 0.$$

(ii) Show that for any closed curve $C$ on the surface $M$, we have

$$\int_C W_{\mathbf{F}} = 0.$$
Problem 7. [17 points; 4, 5, 5, 3.]

Consider the field
\[ \mathbf{F} = (-2xz, x, y^2). \]

Let:
- \( C \) be the circle \( x^2 + y^2 = 1 \) in the plane \( z = 1 \). The circle \( C \) is the intersection of the paraboloid
  \[ z = x^2 + y^2 \]
  and the sphere
  \[ x^2 + y^2 + z^2 = 2; \]
- \( P \) be the piece of the paraboloid below \( C \);
- \( S \) be the spherical cap above \( C \).

Orient \( C \) counterclockwise, and orient \( P \) and \( S \) compatibly.

(i) Find the curl \( \nabla \times \mathbf{F} \) and write down the flux form \( \Phi_{\nabla \times \mathbf{F}} \).
(ii) Evaluate the flux of the curl of $\mathbf{F}$

$$\int\int_P \Phi_{\nabla \times \mathbf{F}}$$

over the piece $P$ of the paraboloid.
(iii) Evaluate the flux of the curl of $F$ over the spherical cap

$$\int \int_S \Phi \mathbf{\nabla} \times \mathbf{F}.$$
(iv) You should have gotten the same answer for (ii) and (iii). Explain why this is expected to hold on general grounds (for all fields F).

Optional Extra Credit: Can you find a second general argument explaining the agreement in (ii) and (iii)?
Problem 8. [12 points.]

Consider the manifold $M \subset \mathbb{C}^2$ consisting of pairs $(z, w) \in \mathbb{C}^2$ such that

$$w = z + e^z.$$

What is the surface area of $M$ contained over the region where $z = x + iy$ satisfies

$$0 \leq x \leq 1, \ 0 \leq y \leq 2\pi.$$
Problem 9. [12 points; 4, 4, 4.]

Fix $a_1, \ldots, a_n > 0$ be positive numbers. Consider

- the ellipsoid $M \subset \mathbb{R}^n$ given by
  \[
  \frac{x_1^2}{a_1^2} + \ldots + \frac{x_n^2}{a_n^2} = 1
  \]
  and oriented by the outward normal.
- the $(n-1)$-form $\omega$ in $\mathbb{R}^n$ given by
  \[
  \omega = x_1 \, dx_2 \wedge dx_3 \wedge \ldots \wedge dx_n - x_2 \, dx_1 \wedge dx_3 \wedge \ldots \wedge dx_n + \ldots + (-1)^{n-1} x_n \, dx_1 \wedge \ldots \wedge dx_{n-1}.
  \]

(i) Compute $d\omega$.

(ii) Express the volume of the solid ellipsoid

\[
\frac{x_1^2}{a_1^2} + \ldots + \frac{x_n^2}{a_n^2} \leq 1
\]

in terms of the volume $\beta_n$ of the unit ball in $\mathbb{R}^n$ and the constants $a_1, \ldots, a_n$. You may use a linear change of coordinates or any other method.
(iii) Determine $\int_M \omega$. 
Problem 10. [12 points; 4, 8.]

It is known that the form
\[ \omega = \frac{ydx - xdy}{x^2 + y^2} \]
is closed on its domain of definition, i.e. \( d\omega = 0 \). \textbf{(You do not need to check this fact.)}

Determine all possible values of the integral
\[ \int_C \omega \]
where \( C \) is an oriented curve without self-intersections joining \((1,0)\) to \((2,-1)\), not passing through the origin.

(i) First determine the integral along a straight line segment \( L \) joining \((1,0)\) to \((2,-1)\).

\textit{Hint: You should try to rewrite the integral in the form}
\[ \int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \left( \frac{u}{a} \right). \]
(ii) Next, consider an arbitrary curve $C$. Compare the integral along $C$ with the answer you found in (i) using Green’s theorem for a suitable region.