Math 31CH - Homework 4. Due Friday, May 2.

1. Solve problem 5.1.1.
2. Solve problem 5.3.1.
3. Solve problem 5.3.8.
4. Read example 5.3.11. Use the same method to solve problem 5.3.10 and 5.3.21.
5. Solve problem 5.3.16.
6. Read example 5.3.10. Use the same method to find the 3-dimensional volume of the 3-dimensional torus in \( \mathbb{R}^6 \) given by the equations

\[
x_1^2 + x_2^2 = a_1^2, \ x_3^2 + x_4^2 = a_2^2, \ x_5^2 + x_6^2 = a_3^2.
\]

Here, \( a_1, a_2, a_3 \) are fixed constants.

7. Fix \( p, q > 0 \). A certain torus \( S \subset \mathbb{R}^4 \) admits the parametrization \( \gamma : [0, 2\pi) \times [0, 2\pi) \to S \) given by

\[
\gamma \left( \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right) = \begin{bmatrix} p \sin \alpha - q \cos \alpha \\ q \sin \alpha + p \cos \alpha \\ \cos \beta \\ \sin \beta \end{bmatrix}.
\]

Find the surface area of the torus.

8. Read example 5.3.13 which computes the \( n \)-dimensional volume of the \( n \)-dimensional sphere \( S^n \) of radius 1 in \( \mathbb{R}^{n+1} \) given by

\[
x_1^2 + \ldots + x_{n+1}^2 = 1
\]

in terms of the volume \( \beta_{n+1} \) of the ball of radius 1 in \( \mathbb{R}^{n+1} \) given by

\[
x_1^2 + \ldots + x_{n+1}^2 \leq 1.
\]

(i) Write a proof of main result

\[
\operatorname{vol}_n (S^n) = (n + 1) \beta_{n+1}
\]

in your own words.

(ii) Record the first 5 values of the volume of the sphere for \( n \leq 5 \).

(iii) Not required. If you like trigonometry and feel inclined to a bit of calculation, solve problem 5.3.15(b). This recovers the \( n = 3 \) case, but with much more work.