This homework is longer and more difficult than the rest. To help you keep up, the topic and the day on which you will have the needed background for each problem is written next to it.

0. Extra credit. (Boundary orientation. Wednesday, May 28.) From Review Exercises, solve problem 6.15(b).

1. (Generalized Stokes. Friday, May 30.) Solve 6.9.1, 6.9.2.


3. (Generalized Stokes. Friday, May 30.) Let $M$ be a compact oriented manifold. Let $\alpha$ and $\beta$ be $k$- and $\ell$-forms respectively. Using the generalized Stokes’ theorem and properties of the exterior derivative, prove the integration by parts formula:

$$\int_M d\alpha \wedge \beta = (-1)^{k+1} \int_M \alpha \wedge d\beta.$$

4. Extra credit. (Green’s theorem. Monday, June 2.) For what simple closed curve $C$ does the line integral

$$\int_C (x^2y + y^3 - y)\,dx + (3x + 2y^2x)\,dy$$

have the largest value? What is that value?

Hints:

- Use Green’s theorem expressing the integral as a double integral over a region $R$ enclosed by $C$.

- To maximize the double integral, imagine the region $R$ to be an expanding membrane emanating from the origin. Is the integrand positive or negative inside of $R$? Is the value of the integral increasing or decreasing as the membrane expands?

5. (Green’s theorem. Monday, June 2.) Consider the field

$$\mathbf{F} = (2xy - 2x^4y)i + (4x + 4x^3y^2 - y^2)j.$$

Let $C$ be any curve in the first quadrant contained in the portion $0 \leq x \leq 1$ starting somewhere on the $y$ axis and ending on the line $x = 1$.

(a) Show that the flux of $\mathbf{F}$ across $C$ is independent of $C$.

Hint: pick two arbitrary curves $C_1$ and $C_2$ as above, and apply Green’s theorem for the flux form $\Phi_{\mathbf{F}}$ and the region between $C_1$ and $C_2$.

(b) Using a convenient path $C$ compute the value of the flux.
6. (Green’s theorem. Monday, June 2.)

Consider the field
\[ F = \frac{x_i + y_j}{x^2 + y^2} \]

(a) Show that \( \nabla \cdot F = 0 \).
(b) Compute the flux of \( F \) across a circle of radius \( a \) centered at the origin.
(c) Show that for any curve \( C \) surrounding the origin, the flux of \( F \) is constant. You may want to repeat the strategy of Problem 5(a).

7. (Green’s theorem and potentials. Monday, June 2.)

Let \( f(x) \) and \( g(y) \) be two continuously differentiable functions defined everywhere on the real line. Consider the field
\[ F = (f(x) + y^2)i + (g(y) + xy)j. \]

(i) Starting from the definitions, show that
\[ \int_C \mathbf{W}_F = 0, \]
where \( C \) is the square of side 1 with corners at \((-1, -1), (1, -1), (1, 1) \) and \((-1, 1)\).
(ii) Can it possibly be true that \( \int_C \mathbf{W}_F = 0 \) for any simple closed curve \( C \)? Please justify your answer.

8. (Potentials in \( \mathbb{R}^2 \). Monday, June 2.) Find the values of \( a, b \) for which the field
\[ F = (ax^2y - 3y - x^2 + 1)i + (x^3 - y - bx - x^2y)j \]
is conservative i.e. the integral of \( \int_C \mathbf{W}_F \) is path independent.

Hint: Recall that this means that the two dimensional curl of \( F \) is zero.

Find a potential function, i.e. a function \( f \) such that \( F = \nabla f \).

9. (Potentials in \( \mathbb{R}^2 \). Monday, June 2.) Consider the field
\[ F = r^n(x_i + y_j). \]

Calculate the (two-dimensional) curl of \( F \). For each value of \( n \) for which curl \( F = 0 \), find a potential \( f \) such that \( F = \nabla f \).

10. (Potentials in \( \mathbb{R}^3 \). Monday, June 2.) Determine the constants \( a, b, c \) for which the field
\[ F = (axy + z^3)i + (x^2 + byz)j + (y^2 + cxz^2 + 1)k \]
for which the integral $\int_C \mathbf{W}_F$ is path independent. Find a potential function for $\mathbf{F}$.

11. (Divergence theorem. Wednesday, June 4.) Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (2 - 2z)\mathbf{k}$. Using the divergence theorem, compute the flux of $\mathbf{F}$ over the surface given by $z = e^{1-x^2-y^2}$, $z \geq 1$, oriented by the upward normal. You may want to replace the surface by a simpler one.

12. (Divergence theorem. Wednesday, June 4.) Check that the divergence theorem is satisfied for the field

$$\mathbf{F} = y\mathbf{i} - (x + 1)\mathbf{j} + z^3\mathbf{k}$$

and the surface $S$ bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 1$.

13. (Stokes’ theorem. Wednesday, June 4.) Verify Stokes’ theorem for the surface $S$ defined by $x^2 + y^2 + 5z = 1$, $z \geq 0$, oriented by the upward normal, and the vector field

$$\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + (x^2 + y^2)\mathbf{k}.$$ That is, compute both sides of Stokes’ theorem and verify that they are equal.

14. (Stokes’ theorem. Wednesday, June 4.) Verify Stokes’ Theorem for the surface defined by $x^2 + y^2 + z^2 = 4$ where $z \leq 0$ oriented by the outward normal, and the field

$$\mathbf{F} = (2y - z)\mathbf{i} + (x + y^2 - z)\mathbf{j} + (4y - 3x)\mathbf{k}.$$ 

15. (Stokes’ theorem. Wednesday, June 4.) Consider the field

$$\mathbf{F} = z^2\mathbf{i} - 2xy\mathbf{j} + z\mathbf{k}.$$ 

(i) Compute the curl of $\mathbf{F}$.

(ii) Show that for any portion $R$ of a sphere of radius $a$ centered at the origin we have

$$\int \int_R \Phi \nabla \times \mathbf{F} = 0.$$ 

(iii) Show that for all simple closed curves $C$ lying on a sphere of radius $a$ centered at the origin, we have

$$\int_C W_{\mathbf{F}} = 0.$$ 

16. (Stokes’ theorem. Wednesday, June 4.) Consider the field

$$\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + z^3\mathbf{k}.$$ 

Let $S$ be the portion of the sphere $x^2 + y^2 + z^2 = 4$ contained above the plane $z = 1$, oriented so that the normal points up. Evaluate both integrals appearing in Stokes’ theorem and check that they are equal.