Problem 1.

Using change of variables, find the area of the first quadrant region bounded by the curves
\[ xy = 2, \ xy = 4, \ xy^3 = 1, \ xy^3 = 2. \]

Solution: We make the change of variables
\[ u = xy, \ v = xy^3 \]
so that
\[ 2 \leq u \leq 4, \ 1 \leq v \leq 2. \]

We have
\[ \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ y^3 & 3xy^2 \end{vmatrix} = 2xy^3 = 2v. \]

Therefore
\[ \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v}. \]

Thus
\[ dx \ dy = \frac{1}{2v} \ du \ dv. \]

We find that the area is
\[ \text{Area} = \int \int dx \ dy = \int_1^2 \int_2^4 \frac{1}{2v} \ du \ dv = \int_1^2 \frac{1}{2v} \ dv \cdot \int_2^4 \ du = \frac{1}{2} \ln v \bigg|_{v=1}^{v=2} \cdot u \bigg|_{u=2}^{u=4} = \ln 2. \]
Problem 2.

Using spherical coordinates, find the mass of the solid with density
\[ \delta = z^3 \]
contained between the sphere
\[ x^2 + y^2 + z^2 = 2 \]
and the cone
\[ z = \sqrt{x^2 + y^2}. \]

Solution: The intersection between the cone and the sphere is
\[ z = 1, \quad x^2 + y^2 = 1. \]
The vertex angle of the cone is \( \frac{\pi}{4} \) with the \( z \)-axis. In spherical coordinates, we have
\[ 0 \leq \rho \leq \sqrt{2}, \quad 0 \leq \phi \leq \frac{\pi}{4}, \quad 0 \leq \theta < 2\pi. \]
The density is
\[ \delta = z^3 = \rho^3 \cos^3 \phi. \]
The integral to be computed is
\[ \int \int \int \delta \, dV = \int_{0}^{1} \int_{0}^{\frac{\pi}{4}} \int_{0}^{2\pi} \rho^3 \cos^3 \phi \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta. \]
Separating variables, the integral becomes
\[ \int_{0}^{\sqrt{2}} \rho^5 \, d\rho \cdot \int_{0}^{\frac{\pi}{4}} \cos^3 \phi \sin \phi \, d\phi \cdot \int_{0}^{2\pi} \, d\theta. \]
We have
\[ \int_{0}^{\sqrt{2}} \rho^5 \, d\rho = \frac{1}{6} \rho^6 \big|_{\rho=0}^{\sqrt{2}} = \frac{4}{3}. \]
Similarly,
\[ \int_{0}^{\frac{\pi}{4}} \cos^3 \phi \sin \phi \, d\phi = -\frac{1}{4} \cos^4 \phi \big|_{\phi=0}^{\frac{\pi}{4}} = \frac{1}{4} \left( 1^4 - \left( \frac{1}{\sqrt{2}} \right)^4 \right) = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}. \]
Finally, \( \int_{0}^{2\pi} \, d\theta = 2\pi \). Putting everything together, the mass becomes
\[ \frac{4}{3} \cdot \frac{3}{16} \cdot 2\pi = \frac{\pi}{2}. \]
Problem 3.

Using cylindrical coordinates, find the volume of the solid between by the sphere

\[ x^2 + y^2 + z^2 = 2 \]

and the paraboloid

\[ z = x^2 + y^2. \]

Solution: The intersection of the sphere and paraboloid can be found by solving the two equations simultaneously:

\[ x^2 + y^2 + z^2 = 2, \quad z = x^2 + y^2 \implies z + z^2 = 2 \implies z = 1, \quad x^2 + y^2 = 1. \]

In cylindrical coordinates, we have

\[ dV = dz \ (r \ dr) \ d\theta. \]

Clearly,

\[ 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi. \]

To find the limits for \( z \), we fix \( r \), and we see that

\[ r^2 \leq z \leq \sqrt{2 - r^2}. \]

The integral is

\[
\int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} dz \ r \ dr \ d\theta = 2\pi \int_0^1 (\sqrt{2-r^2} - r^2) r \ dr = 2\pi \left( \int_0^1 r \sqrt{2-r^2} \ dr - \int_0^1 r^3 \ dr \right) \\
= 2\pi \left( \frac{(2 - r^2)^{3/2}}{-2 \cdot \frac{3}{2}} \bigg|_{r=1} - \frac{1}{4} \right) = 2\pi \left( \frac{2^{3/2} - 1}{3} - \frac{1}{4} \right) = \frac{\pi}{6} \left( 8\sqrt{2} - 7 \right). 
\]
Problem 4.

Calculate the volume of the set $S \subset \mathbb{R}^5$ given by

$$x^2 + y^2 + z^2 + t^2 + w^{2n} \leq 1.$$ 

In order to carry out the calculation, we may need the volume of the unit ball in $\mathbb{R}^4$ computed in class to be $\beta_4 = \frac{\pi^2}{2}$.

**Solution:** Clearly $-1 \leq w \leq 1$. As $w$ is kept fixed, the slice of the solid $S$ for fixed $w$ is simply the ball

$$x^2 + y^2 + z^2 + t^2 \leq 1 - w^{2n}.$$ 

This is a four dimensional ball of radius $(1 - w^{2n})^{\frac{1}{2}}$. The volume of the ball in $\mathbb{R}^4$ of radius $R$ is $\frac{\pi^2}{2} R^4$. For the cross section, the volume is therefore

$$\frac{\pi^2}{2} (1 - w^{2n})^4.$$

The five dimensional volume of $A$ is

$$\frac{\pi^2}{2} \int_{-1}^{1} (1-w^{2n})^2 \, dw = \pi^2 \int_{0}^{1} (1-w^{2n})^2 \, dw = \pi^2 \int_{0}^{1} (1-2w^{2n}+w^{4n}) \, dw = \pi^2 \left( 1 - \frac{2}{2n+1} + \frac{1}{4n+1} \right).$$
Problem 5.

(i) Consider $X \subset \mathbb{R}$ a set of measure 0. Show that $X \times [0, 1]$ is also of measure zero in $\mathbb{R}^2$.

(ii) Let $f : [0, 1] \rightarrow \mathbb{R}$ and $g : [0, 1] \rightarrow \mathbb{R}$ be integrable functions. Let $h : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ be the function with separated variables

$$h(x, y) = f(x) \cdot g(y).$$

Using the integrability theorem stated in class and the properties of sets of measure zero, prove that $h$ is integrable as well.

(iii) Show that the upper Riemann sums are related by

$$U_N(h) = U_N(f) \cdot U_N(g).$$

Conclude from here the separation of variables formula

$$\int_0^1 \int_0^1 h(x, y) \, dx \, dy = \left( \int_0^1 f(x) \, dx \right) \cdot \left( \int_0^1 g(y) \, dy \right).$$

Solution:

(i) Since $X$ is of measure 0, for each $\epsilon > 0$, we can cover $X$ by intervals $B_1, B_2, \ldots, B_k, \ldots$ such that

$$\sum_k \text{vol}_1(B_k) < \epsilon.$$

We can therefore cover $X \times [0, 1]$ by the boxes

$$B_1 \times [0, 1], B_2 \times [0, 1], \ldots, B_k \times [0, 1], \ldots$$

which have total volume

$$\sum_k \text{vol}_2(B_k \times [0, 1]) = \sum_k \text{vol}_1(B_k) \cdot \text{vol}_1([0, 1]) = \sum_k \text{vol}_1(B_k) < \epsilon.$$

This verifies the definition of the fact that $X \times [0, 1]$ has measure zero in $\mathbb{R}^2$.

(ii) If the set of discontinuities for $f$ is denoted by $X$ and the set of discontinuities for $g$ is denoted by $Y$, then by the integrability theorem, $X$ and $Y$ have measure 0 since $f, g$ are integrable.

Now, if

$$h(x, y) = f(x)g(y)$$

is discontinuous at $(x, y)$ then either $f$ is discontinuous at $x$ or $g$ is discontinuous at $y$, because otherwise $f, g$ would be both continuous at $x$ and $y$, hence so will their product. This means that

$$(x, y) \in X \times [0, 1] \cup [0, 1] \times Y.$$

This set has measure zero since it is the union of two sets $X \times [0, 1]$ and $[0, 1] \times Y$ which we know to have measure zero by part (i).
Furthermore, since \( f \) and \( g \) are integrable, \( f \) and \( g \) must be bounded. Thus the product function \( h \) must be bounded as well.

Since \( h \) is bounded and continuous away from a set of measure zero, it must be integrable.

(iii) This problem is proved in the book in Chapter 4.1. We argue that

\[
U_N(h) = U_N(f) \cdot U_N(g).
\]

Indeed, we have

\[
U_N(h) = \sum_C \sup_{(x,y) \in C} h(x,y) \cdot \left(\frac{1}{2^N}\right)^2.
\]

Any dyadic square \( C \) appearing in the sum above can be written as \( C = D \times E \) where \( D, E \) are the two dyadic intervals that give the two sides. Clearly

\[
\sup_{(x,y) \in C} h(x,y) = \sup_{(x,y) \in D \times E} f(x) \cdot g(y) = \sup_{x \in D} f(x) \cdot \sup_{y \in E} g(y).
\]

Thus

\[
U_N(h) = \sum_{D,E} \sup_{x \in D} f(x) \cdot \sup_{y \in E} g(y) \cdot \left(\frac{1}{2^N}\right)^2 = \left(\sum_{D} \sup_{x \in D} f(x) \cdot \frac{1}{2^N}\right) \cdot \left(\sum_{E} \sup_{y \in E} g(y) \cdot \frac{1}{2^N}\right)
\]

\[
= U_N(f) \cdot U_N(g).
\]

Making \( N \to \infty \), we have

\[
\lim_{N \to \infty} U_N(h) = \lim_{N \to \infty} U_N(f) \cdot \lim_{N \to \infty} U_N(g),
\]

or equivalently,

\[
\int_0^1 \int_0^1 h(x,y) \, dx \, dy = \left(\int_0^1 f(x) \, dx\right) \cdot \left(\int_0^1 f(y) \, dy\right).
\]