HOMEWORK 2 SOLUTIONS

4.5.6. (a) Note that the smallest \( n \)-dimensional cube containing the \( n \)-dimensional unit sphere is the \( n \)-fold product

\[ [-1,1] \times \cdots \times [-1,1] \]

which has volume \( 2^n \). We need to show that the sequence \( \frac{\beta_n}{2^n} \) is decreasing. Indeed, this amounts to proving that for \( n \geq 2 \), we have

\[ \frac{\beta_n}{2^n} < \frac{\beta_{n-1}}{2^{n-1}} \iff \beta_n < 2\beta_{n-1}. \]

We have seen in class that

\[ \beta_n = c_n \beta_{n-1}, \]

so it suffices to show that

\[ c_n < 2. \]

We have

\[ c_n = \frac{n-1}{n} c_{n-2} \]

as shown in class, hence

\[ c_n < c_{n-2}. \]

Therefore, for \( n \) odd we have

\[ c_n < c_{n-2} < c_{n-4} < \ldots < c_1 = 2. \]

For \( n \) even,

\[ c_n < c_{n-2} < c_{n-4} < \ldots < c_2 = \frac{1}{2} c_0 = \frac{1}{2} \cdot \pi < 2. \]

We can show even more, namely that the above sequence decreases to 0 i.e.

\[ \lim_{n \to \infty} \frac{\beta_n}{2^n} = 0. \]

To this end, we will consider the even and odd terms separately. For \( n \) even, \( n = 2k \), we have

\[ \frac{\beta_n}{2^n} = \frac{\pi^k}{2^{2k} \cdot k!} = \left( \frac{\pi}{4} \right)^k \cdot \frac{1}{k!} \to 0, \]

where in the limit above we used that \( \frac{\pi}{4} < 1 \), hence \( (\pi/4)^k \to 0 \), while \( k! \to \infty \).

By a similar argument using the formula for \( \beta_n \) for \( n = 2k + 1 \) odd, we have

\[ \frac{\beta_{2k+1}}{2^{2k+1}} = \frac{\pi^k k!}{(2k+1)!} = \frac{\pi^k}{(k+1) \cdots (2k+1)} < \frac{\pi^k}{4^k} \to 0. \]

By the squeeze theorem, the odd terms also converge to 0.
We conclude that
\[ \frac{\beta_n}{2^n} \to 0, \quad \text{as} \quad n \to \infty. \]

(b) \( n = 9 \) (use a calculator)
(c) \( n = 18 \) (use a calculator)
(d) We note that:
\[ \frac{\beta_{2(k+1)}}{\beta_{2k}} = \frac{\pi^{k+1}}{\frac{k!}{(k+1)!}} = \frac{\pi}{k+1} \]
and that \( \frac{\pi}{k+1} < 1 \) for \( k \geq 3 \). In other words, if \( k \geq 3 \), then
\[ \beta_{2(k+1)} < \beta_{2k}, \]
and so the maximum cannot occur at \( \beta_{2k} \) for \( k \geq 4 \). Similarly, using formula for \( \beta_n \) for \( n \) odd, we get that the maximum cannot occur at \( \beta_{2k+1} \) for \( k \geq 3 \). Combining the two cases together gives that the maximum cannot occur at \( \beta_n \) for \( n \geq 7 \). Thus all we must do is evaluate \( \beta_n \) for \( 1 \leq n \leq 6 \) (using a calculator) and determine which is largest; the largest is \( \beta_5 = 5.263789014 \ldots \).

4.5.12. (a) We assume \( a > 0 \). The limits of integration give us the region
\[ D : 0 \leq x \leq a, \quad x^2 \leq y \leq a^2. \]
The region looks like a “right triangle” with vertices \((0,0), (0,a^2)\) and \((a,a^2)\), except that the “hypotenuse” connecting \((0,0)\) to \((a,a^2)\) is part of the parabola \( y = x^2 \). Assuming the integrability hypotheses of Fubini’s Theorem are satisfied (they are), by Fubini’s Theorem the given iterated integral equals
\[ \int \int_D \sqrt{y}e^{-y^2}dA. \]

Side Remark: If the region \( D \) is bounded, and if the function \( f(x,y) \) is bounded and continuous on the interior of \( D \), then the integrability hypotheses of Fubini’s Theorem are satisfied.

(b) The region \( D \) from part (a) can be written as a horizontally simple region:
\[ D : 0 \leq y \leq a^2, \quad 0 \leq x \leq \sqrt{y}. \]

Therefore, by Fubini’s Theorem, the given iterated integral and the above integral are both equal to
\[ \int_0^{a^2} \left( \int_0^{\sqrt{y}} \sqrt{y}e^{-y^2}dx \right)dy. \]
(c) The above integral equals:
\[
\int_0^{a^2} \left[ x \sqrt{y e^{-y^2}} \right]_{x=0}^{\sqrt{7}} dy = \int_0^{a^2} ye^{-y^2} dy = \left[ \frac{-1}{2} e^{-y^2} \right]_{y=0}^{a^2} = \frac{1}{2} \left( 1 - e^{-a^4} \right).
\]

4.5.16. To integrate the absolute value of the function \( f(x, y) \) over a region \( D \), break up \( D \) into two parts: the part where \( f \) is positive and the part where \( f \) is negative, then do each integral separately.

In our case the region \( D \) is the unit square \( 0 \leq x \leq 1, 0 \leq y \leq 1 \). Let:
\[
A = \{(x, y) \in D \mid y - x^2 \geq 0\}, \quad B = \{(x, y) \in D \mid y - x^2 \leq 0\}.
\]
Note that \( A \) is the region from Exercise 4.5.12 with \( a = 1 \), and \( B \) is the remaining part of the unit square. We then have that
\[
\int \int_D \left| y - x^2 \right| dA = \int \int_A \left| y - x^2 \right| dA + \int \int_B \left| y - x^2 \right| dA =
\int \int_A (y - x^2) dA + \int \int_B (x^2 - y) dA.
\]
Note that \( A \) and \( B \) are both vertically simple regions:
\[
A: 0 \leq x \leq 1, \ x^2 \leq y \leq 1, \quad B: 0 \leq x \leq 1, \ 0 \leq y \leq x^2.
\]
Therefore, by Fubini’s Theorem, the desired integral is then equal to
\[
\int_0^1 \int_0^{x^2} (y - x^2) dy dx + \int_0^1 \int_0^{x^2} (x^2 - y) dy dx =
\int_0^1 \int_{x^2}^{1} y^2 - x^2 dy dx + \int_0^1 \int_0^{x^2} x^2 y - \frac{x^2}{2} y^2 dx =
\int_0^1 \left( \frac{1}{2} x^2 - x^4 + \frac{4}{2} x^4 \right) dx + \int_0^1 \left( x^4 - \frac{x^4}{2} \right) dx =
\int_0^1 \left( x^4 - x^2 + \frac{1}{2} \right) dx = \left[ \frac{x^5}{5} - \frac{x^3}{3} + \frac{x^1}{2} \right]_{x=0}^{1} = \frac{11}{30}.
\]

Problem 3. We have
\[
\int \int_D 3y \ dy \ dx = \int_0^1 \int_0^3 3y \ dy dx + \int_1^\frac{\sqrt{7}}{3} \int_0^\frac{\sqrt{7}}{3} 3y \ dy dx = \int_0^1 \left[ \frac{3y^3}{2} \right]_0^{\frac{\sqrt{7}}{3}} dx + \int_1^\frac{\sqrt{7}}{3} \left[ \frac{3y^3}{2} \right]_0^{\frac{\sqrt{7}}{3}} dx =
\int_0^1 \left( \frac{27 - 3x^2}{2} \right) dx + \int_1^\frac{\sqrt{7}}{3} \left( \frac{3 - 3x^3}{2x} \right) dx = \left[ \frac{27x - x^3}{2} \right]_0^{\frac{\sqrt{7}}{3}} + \left[ \frac{3 \ln |x| - x^3}{2x} \right]_1^{\frac{\sqrt{7}}{3}} = 1 + 3 \ln 3.
\]

Problem 4. Reversing the order of integration, we obtain
\[
\int_0^4 \int_0^{\sqrt{4-x}} x \, dy \, dx.
\]

We evaluate the first integral:

\[
\int_0^2 \int_0^{4-y^2} x \, dx \, dy = \int_0^1 \left[ \frac{x^2}{2} \right]_0^{4-y^2} dy = \int_0^2 \frac{16 - 8y^2 + y^4}{2} \, dy = \frac{1}{2} \left[ \frac{240y - 40y^3 + 3y^5}{15} \right]_0^2 = \frac{128}{15}.
\]

Similarly,

\[
\int_0^4 \int_0^{\sqrt{4-x}} x \, dy \, dx = \int_0^4 \left[ \frac{xy}{2} \right]_0^{\sqrt{4-x}} dx = \int_0^4 x\sqrt{4-x} \, dx = \int_0^4 (4-u)\sqrt{u} \, du = \left[ \frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2} \right]_0^4 = \frac{128}{15}.
\]