BUNDLES OF GENERALIZED THETA FUNCTIONS OVER
ABELIAN SURFACES

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Abstract. We study the bundles of generalized theta functions constructed from
moduli spaces of sheaves over abelian surfaces. The splitting type of these bundles is
conjecturally expressed in terms of a new class of semihomogeneous bundles. The con-
jecture is confirmed in degree zero. Fourier-Mukai symmetries of the Verlinde bundles
are found, consistently with strange duality.

1. INTRODUCTION

1.1. Overview. The goal of this paper is to put forward an analogy between aspects
the strange duality proposal for curves and abelian surfaces. Such an analogy is by
no means obvious: we study moduli spaces whose geometries are very different. It is
therefore surprising that the pictures in both cases share common features, some which
we point out below.

In short, our main results are:

(i) we discuss how theta bundles over moduli of sheaves on abelian surfaces depend
on the reference sheaf, analogously to the case of curves considered in [DN];
(ii) we recast the abelian surface strange duality conjecture in terms of Fourier-Mukai
transforms and Verlinde bundles, as in [Po];
(iii) we introduce a new class of semihomogeneous vector bundles, relevant to the
study of the Verlinde bundles;
(iv) we formulate conjectures about the splitting type of the Verlinde bundles, as
done for curves in [O1], [O2];
(v) we confirm our conjectures in degree 0. In particular, we determine the action of
a certain group of torsion points on the space of generalized theta functions; the
curve case was solved in [O2].

We now detail the discussion.

1.2. Moduli of sheaves and their Albanese maps. To set the stage, consider a
complex polarized abelian surface \((A, H)\), and fix the Mukai type \(v\) of sheaves \(E \to A\),
so that

\[ v = \text{ch}(E). \]

The Mukai vectors used in this paper are of the form

\[ v = (r, kH, \chi), \]

and the following assumption will be made throughout:

(A.1) the vector \( v = (r, kH, \chi) \) is primitive, and furthermore the Mukai self pairing

\[ d_v = \frac{1}{2} \langle v, v \rangle := k^2 \cdot \frac{H^2}{2} - r\chi \]

is an odd positive integer.

We consider the moduli space \( M_v \) of \( H \)-semistable sheaves of topological type \( v \). The moduli space comes equipped with the Albanese morphism

\[ \alpha = (\alpha^+, \alpha^-) : M_v \to A \times \hat{A}, \]

which, up to the choice of a reference sheaf \( E_0 \) of type \( v \), takes sheaves \( E \) to their determinant and determinant of the Fourier-Mukai transform

\[ \alpha(E) = (\det \mathcal{R}\mathcal{S}(E) \otimes \det \mathcal{R}\mathcal{S}(E_0)^\vee, \det E \otimes \det E_0^\vee). \]

Here

\[ \mathcal{R}\mathcal{S} : \mathcal{D}(A) \to \mathcal{D}(\hat{A}) \]

denotes the Fourier-Mukai transform. The Albanese fiber will be denoted by \( K_v \), thus parametrizing semistable sheaves with fixed determinant and fixed determinant of their Fourier-Mukai transform. It is known that when \( (A, H) \) is generic, \( K_v \) is a holomorphic symplectic manifold of dimension \( 2d_v - 2 \), deformation equivalent to the generalized Kummer variety of the same dimension \([Y1]\).

1.3. Theta bundles. We consider the natural theta line bundles over the above moduli spaces. Assume that \( w \) is a Mukai vector orthogonal to \( v \) in the sense that in \( K \)-theory we have

\[ \chi(v \cdot w) = 0. \]

Pick a representative \( F \to A \) of the Mukai vector \( w \), and construct the Fourier-Mukai transform of \( F \) with kernel the universal sheaf \( \mathcal{E} \to M_v \times A: \]

\[ \Theta_w = \det \mathcal{R}\mathcal{P}(\mathcal{E} \otimes q^* F)^{-1}. \]

Generalized theta functions are sections of \( \Theta_w \) over either of the moduli spaces \( K_v \) or \( M_v \) considered above.

\[ ^1 \text{or by descent from the Quot scheme in the absence of the universal sheaf} \]
1.4. **Relating different theta bundles.** The notation $\Theta_w \to \mathfrak{M}_v$ is slightly imprecise, since in our context the bundles $\Theta_F$ may depend on the choice of representative $F$. For the purposes of the introduction, this ambiguity will be ignored\(^2\). The following result, paralleling the Drézet-Narasimhan theorem for bundles over curves [DN], controls this imprecision:

**Theorem 1.** For two sheaves $F_1$ and $F_2$ of the same Mukai vector orthogonal to $v$ we have

$$\Theta_{F_1} = \Theta_{F_2} \otimes \left( (-1) \circ \alpha^+ \right)^* \left( \det F_1 \otimes \det F_2^{-1} \right) \otimes \left( (-1) \circ \alpha^- \right)^* \left( \det \mathcal{R} \mathcal{S}(F_1) \otimes \det \mathcal{R} \mathcal{S}(F_2)^{-1} \right).$$

1.5. **Verlinde numbers.** The holomorphic Euler characteristics of the bundles $\Theta_w$ are calculated in [MO]. For instance,

$$\chi(K_v, \Theta_w) = \frac{d_v^2}{d_v + d_w} \left( \frac{d_v + d_w}{d_v} \right).$$

In order to use these numerics for the study of generalized theta functions, one needs to prove that

$$h^0(K_v, \Theta_w) = \chi(K_v, \Theta_w).$$

This occurs for instance when $\Theta_w$ carries no higher cohomology over $K_v$, or alternatively over some smooth birational model of $K_v$. By [BMOY], this is the case under general assumptions, for instance for primitive Mukai vectors of the form

(A.2) $v = (r, kH, \chi)$, $w = (r', k'H, \chi')$ with $k, k' > 0$ and $\chi, \chi' < 0$.

The less explicit but more general assumption

(A.2') $\Theta_w$ belongs to the movable cone of $K_v$

is however what is truly used in our proofs.

For obvious reasons, we prefer (A.2)' to (A.2) when stating the main calculation of this paper, contained in Sections 5.3 and 5.4 and reviewed in Sections 1.8 and 1.9. The easier to parse (A.2) should otherwise do the job for all other results.

1.6. **Bundles of generalized theta functions.** When assumptions (A1) – (A2) are satisfied, we push forward $\Theta_w$ via the morphism $\alpha$. The resulting bundles of generalized theta functions encode dependence on the determinant and determinant of the Fourier-Mukai. We obtain in this fashion the Verlinde bundles

$$E(v, w) = \alpha_* \Theta_w$$

\(^2\text{we refer the reader to Convention 1 for the precise conventions on theta bundles}\)
over the abelian four-fold \( A \times \hat{A} \). Note that by (1), its rank equals
\[
\frac{d_v^2}{d_v + d_w} \left( \frac{d_v + d_w}{d_v} \right).
\]

In the context of curves, the analogous bundles were introduced and studied in [Po], and were further analyzed in [O1], [O2].

Here, we formulate conjectures about the Verlinde bundles for abelian surfaces, and prove them in several general cases. In particular, we point out how the techniques of [O1] and [O2] need to be adapted to the new setup we consider.

1.7. **Semihomogeneous bundles.** Assume \((A, \Theta)\) is a principally polarized surface\(^3\), such that \(\Theta\) is symmetric
\[
(-1)^*\Theta = \Theta.
\]

We showed in [O2], in the context of a calculation for curves, that for any pair of coprime odd integers \((a, b)\) there exists a unique symmetric semihomogeneous bundle \(W_{a,b}\) over \(A\) such that
\[
\text{rank } W_{a,b} = a^2 \quad \text{and} \quad \det W_{a,b} = \Theta^{ab}.
\]

Recall that semihomogeneity is the requirement that all translations of \(W_{a,b}\) by \(x \in A\) are of the form
\[
t^x W_{a,b} = W_{a,b} \otimes y
\]
for some line bundle \(y\) over \(A\).

We may also consider semihomogeneous bundles over the dual abelian variety \((\hat{A}, -\hat{\Theta})\), where \(-\hat{\Theta} = \det R_S(\Theta)^{-1}\) is the dual polarization. We will use the notation \(W_{a,b}^\dagger\) for the corresponding bundles of rank \(a^2\) and determinant \(\hat{\Theta}^{ab}\).

A more general class of semihomogeneous vector bundles
\[
W(P) \to A \times \hat{A},
\]
depending on a triple \(P\) of rational numbers, will be introduced and studied in Section 3. For instance, for a triple written in lowest terms
\[
P = \left( \frac{b}{a}, \frac{d}{c}, h \right),
\]
where \((a, c)\) are odd and coprime and \(h \in \mathbb{Z}\), we have
\[
W(P) = W_{a,b} \otimes W_{c,d}^\dagger \otimes P^h.
\]

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\(^3\)Most of our results also hold without change for non principal polarizations. We focus on principal polarizations to keep the numerics simple.
Here $\mathcal{P}$ is the normalized Poincaré bundle over $A \times \hat{A}$. For general triples $\mathcal{P}$, the bundles $W(\mathcal{P})$ do not admit such simple expressions. The bundles $W(\mathcal{P})$ are crucial for our study of the Verlinde bundles.

1.8. **Action of torsion points on generalized theta functions.** We will now consider the case $c_1(v) = 0$ i.e.

$$v = (r, 0, \chi),$$

with $r$ and $\chi$ odd coprime integers, cf. (A.1). The dual vector $w$ must have the form

$$w = (rh, k\Theta, -\chi h)$$

for some integers $h$ and $k$. Since the strict inequalities in (A.2) are now not fulfilled, under the given numerics we make the weaker assumption (A.2)' requiring that $\Theta_w$ belong to the movable cone of $K_v$. \footnote{This is achieved for sufficiently large slope $\mu(w) > \mu^+$. By the argument in Section 6 of \cite{BMOY}, an explicit bound is given by\hspace{1cm}}

We consider the action of $(x, y) \in A \times \hat{A}$ on the moduli space of sheaves $\mathfrak{M}_v$ given by

$$(x, y) : E \mapsto t_x^*E \otimes y.$$ 

The action leaves $K_v$ invariant provided $\chi x = 0$, $ry = 0$. Let us write

$$\gcd(\chi, k) = a, \gcd(r, k) = b.$$ 

If the stronger condition

$$ax = 0, \quad by = 0$$

is satisfied, the action lifts to the line bundle $\Theta_w \to K_v$. We will assume this is the case. We show

**Theorem 2.** *If $\zeta = (x, y)$ has order $\delta$, then the trace of $\zeta$ on the space of generalized theta functions equals*

$$\text{Trace} \left( \zeta, H^0(K_v, \Theta_w) \right) = \frac{d_v^2}{d_v + d_w} \left( \frac{d_v/\delta + d_w/\delta}{d_v/\delta} \right).$$

We believe the Theorem should hold for arbitrary $c_1(v)$, but we are unable to prove this.
1.9. Explicit expressions for the Verlinde bundles. Still under the assumption of Subsection 1.8, we obtain an explicit expression for the Verlinde bundles. The situation is easily understood when

\[(r, k) = (\chi, k) = 1.\]

Then

\[E(v, w) = \bigoplus \left( W_{-\chi, k} \otimes W_{-r, -k}^\dagger \right) \otimes P^{-h}.\]

This much can easily be derived from the representation theory of Heisenberg groups.

The difficulty of the calculation lies however in the case when the integers \(k\) and \(r\chi\) are not coprime. Representation theory only gives \(E(v, w)\) up to torsion line bundles over \(A \times \hat{A}\) of orders dividing \((a, b)\). We will prove the following:

**Theorem 3.** We have

\[E(v, w) = \left( W_{-\chi, a} \otimes W_{r, -b}^\dagger \right) \otimes P^{-h} \otimes \bigoplus_{\zeta} \ell_{m_\zeta}.\]

The sum is taken over line bundles \(\zeta\) over \(A \times \hat{A}\) of orders dividing \((a, b)\). A line bundle \(\zeta\) of order exactly \(\omega\) comes with multiplicity

\[m_\zeta = \frac{1}{d_v + d_w} \sum_{\delta | ab} \frac{\delta^4}{(ab)^2} \left\{ \frac{ab/\omega}{\delta} \right\} \left( \frac{d_v/\delta + d_w/\delta}{d_v/\delta} \right).\]

The line bundles \(\ell \to A \times \hat{A}\) in the sum are roots of \(\zeta\) of order \((\frac{\chi}{a}, \frac{r}{b})\). For each \(\zeta\), only one such root \(\ell\) is chosen.

The Jordan totient \{(\ )\} appearing above is defined in terms of prime factorization. Specifically, for any integer \(h \geq 2\), we decompose

\[h = p_1^{a_1} \cdots p_n^{a_n}\]

into powers of primes. We set

\[\left\{ \lambda \atop h \right\} = \begin{cases} 0 & \text{if } p_1^{a_1-1} \cdots p_n^{a_n-1} \text{ does not divide } \lambda, \\ \prod_{i=1}^n \left( \epsilon_i - \frac{1}{p_i} \right) & \text{otherwise}, \end{cases}\]

where

\[\epsilon_i = \begin{cases} 1 & \text{if } p_i^{a_i} | \lambda, \\ 0 & \text{otherwise.} \end{cases}\]

The symbol is set to 1 if \(h = 1\).

A description of the Verlinde bundles in arbitrary generality is proposed in Conjecture 2 of Section 4. The semihomogeneous bundles \(W(P)\) of Section 3 enter the conjectured expression.
1.10. **Fourier-Mukai symmetries.** We furthermore prove several symmetries of the Verlinde bundles $E(v, w)$ over the abelian four-fold $A \times \hat{A}$. We show

**Theorem 4.** Assuming that $c_1(v)$ and $c_1(w)$ are divisible by their ranks $r$ and $r'$, we have

$$E(v, w)^\vee \cong \hat{E}(w, v).$$

The isomorphism (3) is obtained by direct comparison of both sides, using Theorem 3. The same result should be true for any vectors $v$ and $w$ satisfying (A.1) – (A.2), and in fact, it is implied by Conjecture 2.

1.11. **Strange duality.** The above symmetry of the Verlinde bundles is related to the strange duality conjecture\(^5\). This was observed in the case of curves by Popa [Po].

Considering the fibers of (3) over the origin, we obtain isomorphic spaces

$$H^0(K_v, \Theta_w)^\vee \cong H^0(M_w, \Theta_v).$$

As stated, this is merely saying that the dimensions of both vector spaces agree. However, there is a geometrically induced map, called strange duality, which conjecturally yields the isomorphism above. Even stronger, various strange duality maps can be packaged into an explicit bundle morphism, constructed as a corollary of Theorem 1

$$\text{SD} : E(v, w)^\vee \rightarrow \hat{E}(w, v).$$

In many cases, SD should provide a geometric isomorphism of bundles, as predicted by Theorem 4. This isomorphism is proven generically in [BMOY] for an infinite class of topological types, but the general case is still open.

1.12. **Comparison to the case of curves.** To end this introduction, let us remark that a similar picture emerges in the case of curves, cf. [O2]. Indeed, let $C$ be smooth of genus $g$, and consider the vectors

$$v = r [\mathcal{O}_C], \quad w = k [\kappa],$$

for a symmetric Theta characteristic $\kappa$. The bundles of rank $r$, level $k$ generalized theta functions

$$E_{r, k} = \text{det}_* \left( \Theta^k_{\kappa} \right) \rightarrow \text{Jac}(C)$$

can be obtained pushing forward the pluri-theta bundles $\Theta_w = \Theta^k_{\kappa} \rightarrow \mathcal{M}_v$ via the determinant map

$$\text{det} : \mathcal{M}_v \rightarrow \text{Jac}(C).$$

\(^5\)for surfaces, strange duality phenomena have first been studied by Le Potier [LP2]
The pushforwards $E_{r,k}$ take the form

$$E_{r,k} = W_{r,k} \oplus \bigoplus_{\zeta} \ell^{m_{\zeta}(r,k)}.$$  

The torsion line bundles $\zeta$ have orders dividing $a = \gcd(r,k)$ and the multiplicities $m_{\zeta}(r,k)$ of their $\frac{r}{a}$-roots $\ell$ are explicit. Furthermore, the symmetry

$$E_{r,k}^\vee \cong \hat{E}_{k,r}$$

is a manifestation of strange duality.

1.13. **Outline.** The paper is organized as follows. The next section discusses preliminary results about theta bundles and their behavior under étale pullbacks; in particular Theorem 1 is proved there. The third section introduces and studies a new class of semi-homogeneous vector bundles relevant for strange duality; this section can also be read independently of the rest of the paper. Section 4 explains the conjectural expression of the Verlinde bundles. The fifth section contains the computation confirming Theorem 2. This is the main calculation of the paper, and Theorems 3 and 4 follow from it.

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2. **Preliminaries on theta bundles**

This section collects various observations about theta bundles. The main result here is Theorem 1, which describes how the theta bundles vary with respect to the reference sheaf. Over curves, a similar statement was made in [DN], and proved by entirely different methods. As a corollary of the theorem, we construct the strange duality map between the Verlinde bundles.

2.1. **Setup.** Let $(A, \Theta)$ be a principally polarized abelian surface, with $\Theta$ symmetric. Throughout the paper, we will use the usual conventions on the Fourier-Mukai transform $R_S : D(A) \to D(\hat{A})$, $R_S(E) = R_p(\mathcal{P} \otimes q^*E)$, where $\mathcal{P}$ is the normalized Poincaré bundle over $A \times \hat{A}$. Dually, we consider the transform $R_{\hat{S}} : D(\hat{A}) \to D(A)$.

If $E$ satisfies the index theorem, we often write $\hat{E}$ for the sheaf representing $R_S(E)$, up to shift. Furthermore, we make frequent use of the following identities [M1]:

$$R_S(t^*_x E) = R_S(E) \otimes \mathcal{P}_{-x},$$

$$R_S(E \otimes y) = t^*_y R_S(E).$$
We set
\[ \hat{\Theta} = \det R_S(\Theta), \]
so that \(-\hat{\Theta}\) is the polarization on the dual abelian variety \(\hat{A}\). Finally, we write
\[ \Phi : A \to \hat{A}, \quad \hat{\Phi} : \hat{A} \to A \]
for the morphisms induced by \(\Theta\) and \(\hat{\Theta}\), so that
\[ \Phi \circ \hat{\Phi} = -1, \quad \hat{\Phi} \circ \Phi = -1. \]

Consider two orthogonal Mukai vectors
\[ v = (r, k\Theta, \chi), \quad w = (r', k'\Theta, \chi'), \]
satisfying assumptions \((A.1)\) and \((A.2)\). Central for our arguments is the following diagram \([Y1], [MO]\):

\[
\begin{array}{ccc}
K_v \times A \times \hat{A} & \xrightarrow{\tau} & \mathcal{M}_v \\
\downarrow p & & \downarrow \alpha \\
A \times \hat{A} & \xrightarrow{\Psi} & A \times \hat{A}
\end{array}
\]

We now explain the notation. The morphism
\[ \tau : K_v \times A \times \hat{A} \to \mathcal{M}_v \]
is given by
\[ \tau(E, x, y) = t_x^* E \otimes y, \]
and the Albanese map \(\alpha\) is defined as
\[ \alpha(E) = \left( \det R_S(E) \otimes \hat{\Theta}^{-k}, \det E \otimes \Theta^{-k} \right). \]
Recall that we write \(K_v\) for the fiber of \(\alpha\) over the origin. Lemma 4.3 of \([Y1]\) identifies the horizontal morphism
\[ \Psi(x, y) = (-\chi x + k\hat{\Phi}(y), k\Phi(x) + ry). \]

Lemma 15 of the Appendix shows that \(\Psi\) has degree \(d_v^4\).

2.2. Properties of the theta bundles. As already mentioned, we prove the following analogue of the Drézet-Narashimhan theorem \([DN]\), originally conjectured in \([MO]\):

**Theorem 1.** If \(F_1, F_2\) are sheaves of the same Mukai vector orthogonal to \(v\), then
\[ \Theta_{F_1} = \Theta_{F_2} \otimes ((-1) \circ \alpha^+) \ast (\det F_1 \otimes (\det F_2)^\vee) \otimes ((-1) \circ \alpha^-) \ast (\det R_S(F_1) \otimes \det R_S(F_2)^\vee). \]
Proof. We begin by noting that $\Theta_F$ depends \textit{a priori} on the holomorphic $K$-theory class of the sheaf $F$. To compare different theta bundles, we consider the virtual difference

$$f = F_1 - F_2.$$ 

By Lemma 1 of [MO], we may assume that $K$-theoretically we have

$$f = M - O + O_Z - O_W,$$

for a line bundle $M$ over $A$ of degree 0, and subschemes $Z, W$ of the same length $\ell$. This gives

$$\det f = M, \quad \det R S(f) = \hat{M} \otimes P_{a(Z) - a(W)},$$

where $a$ is the addition morphism, and

$$\hat{M} = \det R S(M)$$

stands for the determinant of the Fourier-Mukai transform. In fact, since $M = P_y$ for some $y \in \hat{A}$, we have $\hat{M} = O$.

Let us write

$$m = M - O, \quad n = O_Z - O_W,$$

and observe that

$$\Theta_f = \Theta_m \otimes \Theta_n.$$ 

We show

$$\Theta_n \cong (\alpha^-)^* P_{a(W)} - a(Z).$$

It is enough to check this isomorphism along any test family of sheaves. Indeed, consider a flat family

$$\mathcal{E} \to S \times A$$

of sheaves of type $\nu$, inducing a morphism

$$\alpha^- : S \to \hat{A}$$

by taking determinants and twisting. Remark that by the see-saw theorem

$$\det \mathcal{E} = (\alpha^- \times 1)^* P \otimes p^* \mathcal{V} \otimes q^* \Theta^k,$$

for some line bundle $\mathcal{V} \to S$, with $p, q$ being the projections from $S \times A$. We calculate

$$\det R p_!(\mathcal{E} \otimes q^* O_Z) = \bigotimes_{z \in Z} \det \mathcal{E}_z = \bigotimes_{z \in Z} ((\alpha^-)^* P_z \otimes \mathcal{V}) = (\alpha^-)^* P_{a(Z)} \otimes \mathcal{V}^\ell.$$ 

Therefore,

$$\Theta_n = \left((\alpha^-)^* P_{a(Z)} \otimes \mathcal{V}^\ell\right)^{-1} \otimes \left((\alpha^-)^* P_{a(W)} \otimes \mathcal{V}^\ell\right) = (\alpha^-)^* P_{a(W) - a(Z)},$$

as claimed.
It remains to prove that
\[ \Theta_m = (\alpha^+) \star M^{-1}. \]
First, we will verify this equality after pullback by \( \tau \). To begin, we calculate the pullback of the right hand side
\[ \tau^*(\alpha^+) \star M^{-1} = (-\chi x + k\Phi(y)) \star M^{-1} = (M^x \boxtimes \Phi^* M^{-k}). \]
We claim next that
\[ \tau^* \Theta_m = Q \boxtimes (M^x \boxtimes \Phi^* M^{-k}) \]
for some line bundle \( Q \) over \( K_v \), where \( Q \) is the restriction of \( \tau^* \Theta_m \) to \( K_v \). The Chern class of \( Q \) depends only on the Chern class of \( M \), and since for \( M = O \) we obtain the trivial bundle, this must be the case for all \( M \)'s of degree zero. Since \( K_v \) is simply connected, \( Q \) is trivial.

To identify the remaining term, note that the restriction of \( \tau^* \Theta_m \) to \( \{E\} \times A \times \hat{A} \) equals
\[ \mathcal{L}_E = \det R^{p_{23!}}(m_{12}^* E \otimes p_{13}^* P \otimes p_1^* (M - O))^{-1}, \]
where
\[ m_{12} : A \times A \times \hat{A} \to A \]
denotes the addition on the first two factors, and the \( p \)'s are the projections. We prove that
\[ \mathcal{L}_E = M^x \boxtimes \Phi_D^* M^{-1}, \]
where \( D \) is the determinant of \( E \), and
\[ \Phi_D : A \to \hat{A}, \quad \Phi_D : \hat{A} \to A \]
are the induced morphisms. The idea of the proof is already contained in the above argument. We first note that \( \mathcal{L}_E \) only depends on the holomorphic \( K \)-theory class of \( E \). In fact, we argue that \( \mathcal{L}_E \) depends on the rank, determinant and Euler characteristic of \( E \). Indeed, for two sheaves \( E_1, E_2 \) with the same data as above, we form the virtual difference
\[ e = E_1 - E_2. \]
By Lemma 1 of [MO], we can write
\[ e = O_Z - O_W, \quad \text{with } \ell(Z) = \ell(W). \]
Then
\[ \mathcal{L}_e = \mathcal{L}_{E_1} \otimes \mathcal{L}_{E_2}^{-1} \]
is trivial, since for any \( a \in A \), we have

(9) \[
\det R_{p_{23}}(m_{12} \mathcal{O}_{a} \otimes p_{13}^{*} \mathcal{P} \otimes p_{1}^{*} (\mathcal{M} - \mathcal{O})) = \det ((t_{a} \times 1) \circ (-1 \times 1))^{*} (\mathcal{P} \otimes p_{1}^{*} (\mathcal{M} - \mathcal{O}))
\]
\[= \mathcal{M}^{-1}.
\]

Since for fixed rank and Euler characteristic, \( L_{E} \) depends only on the determinant of \( E \), we may assume that \( E \) splits. Since both sides of (8) are multiplicative in \( E \), it therefore suffices to consider the cases

\[ E = \mathcal{O}, \; E = \mathcal{O}(D), \; E = \mathcal{O}_{o}. \]

Now, note that for \( E = \mathcal{O} \), we have

\[ L_{\mathcal{O}} = \det R_{p_{23}}(m_{12} \mathcal{O} \otimes p_{13}^{*} \mathcal{P} \otimes p_{1}^{*} (\mathcal{M} - \mathcal{O}))^{-1} = \tilde{\mathcal{M}}^{-1} \cong \mathcal{O}, \]

while for \( E = \mathcal{O}_{o} \) we obtain by (9)

\[ L_{\mathcal{O}_{o}} = \det R_{p_{23}}(m_{12} \mathcal{O}_{o} \otimes p_{13}^{*} \mathcal{P} \otimes p_{1}^{*} (\mathcal{M} - \mathcal{O}))^{-1} \cong \mathcal{M}. \]

The calculation for \( E = \mathcal{O}(D) \) is more involved. We need to show

\[ L = \mathcal{M}^{x(D)} \otimes \tilde{\Phi}_{D}^{*} \mathcal{M}^{-1}. \]

Observe that over \( A \times A \) we have

\[ m_{12}^{*} \mathcal{O}(D) = (1 \times \tilde{\Phi}_{D})^{*} \mathcal{P} \otimes p_{1}^{*} \mathcal{O}(D) \otimes p_{2}^{*} \mathcal{O}(D). \]

We compute

\[
L^{-1} = \det R_{p_{23}!} (p_{12}^{*} (1 \times \tilde{\Phi}_{D})^{*} \mathcal{P} \otimes p_{13}^{*} \mathcal{P} \otimes p_{1}^{*} (\mathcal{M}(D) - \mathcal{O}(D)))
\]
\[= (\Phi_{D} \times 1)^{*} \det R_{p_{23}!} (p_{12}^{*} \mathcal{P} \otimes p_{13}^{*} \mathcal{P} \otimes p_{1}^{*} (\mathcal{M}(D) - \mathcal{O}(D))).\]

Note furthermore that

\[ p_{12}^{*} \mathcal{P} \otimes p_{13}^{*} \mathcal{P} = (1 \times \tilde{m})^{*} \mathcal{P} \]

where

\[ 1 \times \tilde{m} : A \times \hat{A} \times \hat{A} \to A \times A \]

is the addition map. We conclude

\[
L^{-1} = (\Phi_{D} \times 1)^{*} \tilde{m}^{*} \det R_{p_{23}!} (\mathcal{P} \otimes p_{1}^{*} (\mathcal{M}(D) - \mathcal{O}(D)))
\]
\[= (\Phi_{D} \times 1)^{*} \tilde{m}^{*} \left( \det \tilde{\mathcal{M}}^{x} \otimes \tilde{D} \otimes \det \tilde{D}^{-1} \right)
\]
\[= (\Phi_{D} \times 1)^{*} \tilde{m}^{*} \left( t_{\mathcal{M}}^{*} \det \tilde{D} \otimes \det \tilde{D}^{-1} \right)
\]
\[= (\Phi_{D} \times 1)^{*} \tilde{m}^{*} \tilde{\Phi}_{D}(\mathcal{M}).\]

Now for a degree 0 line bundle \( U \) we have

\[ \tilde{m}^{*} U = p_{1}^{*} U \otimes p_{2}^{*} U, \]
hence
\[ L^{-1} = (\Phi_D \times 1)^* \left( \Phi_D (\mathcal{M}) \boxtimes \Phi_D (\mathcal{M}) \right) = (\Phi_D \times 1)^*(\Phi_D^* \mathcal{M} \boxtimes \Phi_D^* \mathcal{M}) = \mathcal{M}^{-\chi(D)} \boxtimes \Phi_D^* \mathcal{M} \]
as claimed in (8).

Equality (7) is now checked under pullback by \( \tau \). It remains to observe that the assignment
\[ \mathcal{M} \to \Theta_m \otimes (\alpha^+)^* \mathcal{M} \]
defines a morphism
\[ \pi : \hat{A} \to \text{Pic}(\mathcal{M}_v). \]
This amounts to constructing a universal line bundle
\[ \mathcal{U} \to \mathcal{M}_v \times \hat{A} \]
which restricts to \( \Theta_{\mathcal{M}_v} \otimes (\alpha^+)^* \mathcal{M} \) over \( \mathcal{M}_v \times \{\mathcal{M}\} \). Indeed, it suffices to define
\[ \mathcal{U} = \det \mathcal{R}p_{23!}(p_{13}^* \mathcal{P} \otimes p_{11}^* (\mathcal{M} - \mathcal{O}) \otimes p_{12}^* \mathcal{E})^{-1} \otimes (\alpha^+ \times 1)^* \mathcal{P}, \]
using the pushforward from
\[ A \times \mathcal{M}_v \times \hat{A} \to \mathcal{M}_v \times \hat{A}, \]
for a universal sheaf \( \mathcal{E} \to A \times \mathcal{M}_v \). A quasi-universal sheaf may be used if necessary. Note however that the above discussion implies that
\[ \tau^* \circ \pi = 0. \]
Since the kernel of \( \tau^* \) is discrete, \( \pi \) must be constant. Since \( \pi(\mathcal{O}) = \mathcal{O} \), we must have \( \pi(\mathcal{M}) = \mathcal{O} \) completing the proof. \( \square \)

**Convention 1.** The theorem above shows that \( \Theta_F \) only depends on the rank, Euler characteristic, determinant and determinant of the Fourier-Mukai of the bundle \( F \). The Mukai vectors used in this paper are all of the form
\[ w = (r', k', \Theta, \chi'). \]
We make the convention that all Theta bundles
\[ \Theta_w := \Theta_F \to \mathcal{M}_v \]
are calculated with respect to \( K \)-classes \( F \) with
\[ \text{rank } F = r', \chi(F) = \chi', \det F = \Theta^{k'}, \det \mathcal{R}S(F) = \widehat{\Theta}^{k'}. \]
Example 1. Assume that $v = (1, 0, -n)$ so that $\mathcal{M}_v \cong A^{[n]} \times \widehat{A}$ via the isomorphism 

$$(Z, y) \mapsto I_Z \otimes y.$$ 

Then, $\alpha$ corresponds to the morphism 

$$(-a, 1): A^{[n]} \times \widehat{A} \to A \times \widehat{A},$$

where as usual $a$ is the addition map. For a sheaf $F \to A$ of rank $r$, we obtain 

$$\Theta_F = \det R p_{12!} (p^*_{12} \mathcal{O}_Z \otimes p^*_2 E \otimes p^*_3 P)\bigotimes \det R S (F)^{-1} \otimes (a, 1)^* \mathcal{P}^r.$$

The second line bundle can be found via the see-saw theorem and Section 5 of [EGL]

$$\Theta_F = \det R p_{12!} (p^*_{12} F \otimes p^*_2 P) \otimes \det R p_{12!} (p^*_{12} (\mathcal{O}_Z \otimes p^*_2 F \otimes p^*_3 P))^{-1},$$

The line bundle $\Theta_w$ was shown to be independent of choices on the simply connected manifold $K_v$. Furthermore, we proved that 

$$\tau^* \Theta_w = \Theta_w \boxtimes \mathcal{L}.$$ 

The line bundle $\Theta_w$ was shown to be independent of choices on the simply connected manifold $K_v$. Furthermore, we proved that 

$$\mathcal{L} = \det R p_{23!} \left( m_{12}^* E \otimes p^*_{13} P \otimes p^*_3 F \right)^{-1},$$

with the $p$'s denoting the projections on the corresponding factors of $A \times A \times \widehat{A}$, while 

$$m_{12} : A \times A \times \widehat{A} \to A$$
is as usual the addition on the first two factors. We moreover calculated the Euler characteristic of $L$, cf. [MO]. In the lemma below, we identify $L$ explicitly. As a corollary we obtain

$$\Psi^*E(v, w) = p_\bullet \tau^* \Theta_{w} = H^0(K_v, \Theta_{w}) \otimes L.$$  

We will use this equation in Section 5.

**Lemma 1.** For any two sheaves $E$ and $F$ satisfying Convention 1, we have

$$L = (\Theta - \chi'k - \chi k' \otimes \hat{\Theta} - r k' - r'k) \otimes P^r\chi + kk'.$$

**Proof.** The proof is similar to that of Theorem 1. First, we note that the bundle $L_{E,F}$ depends only on the holomorphic $K$-theory classes of $E$ and $F$. In fact, we will furthermore remark below that the line bundle only depends on the Mukai vectors $v$ and $w$, the determinant and determinant of the Fourier-Mukai of $E$ and $F$. In other words, we argue that

$$L_{E,F} \cong L_{E^*, F}, \quad L_{E,F} \cong L_{E,F'}$$

for pairs $(E, E^*)$ and $(F, F')$ of the same Mukai vectors, determinants and determinants of Fourier-Mukai.

In fact, only the statement about the first argument $E$ will be useful to us. For this isomorphism, consider the virtual sheaf $e = E - E^*$.  

Note that in $K$-theory, we have

$$e = \mathcal{O}_Z - \mathcal{O}_W$$

for two zero-dimensional subschemes which have the same length [MO], and since the Fourier-Mukai transform of $e$ has trivial determinant, we must have $a(Z) = a(W)$. We prove

$$L_{e,F} = L_{E,F} \otimes L_{E^*,F}^{-1}$$

is trivial. Over $A \times \hat{A}$, we calculate

$$\det \mathbf{R}p_{23!}(m_{12}\mathcal{O}_Z \otimes p_{13}^* \mathcal{P} \otimes p_1^* \mathcal{F}) \cong \bigotimes_{z \in Z}((t_z \times 1) \circ (-1 \times 1))^* \det (\mathcal{P} \otimes p_1^* \mathcal{F})$$

which only depends on $a(Z)$ by the theorem of the square. Thus, we obtain the same answer replacing $Z$ by $W$, therefore showing $L_{e,F}$ is trivial. The argument for the pair $(F, F')$ is similar.
With this understood, we prove the lemma. We may assume then that $E$ splits as a direct sum of copies of $O$, $\Theta$ and structure sheaves $O_\alpha$. In fact, since $L_{E,F}$ is multiplicative in $E$, it suffices to prove the lemma separately for the three sheaves $E = O$, $E = \Theta$ and $E = O_\alpha$.

First, for $E = O$, we obtain
$$L = \det R_{p_23!}(m_{12}O \otimes p_{13}^*P \otimes p_1^*F)^{-1} \cong O \boxtimes \left(\det \tilde{F}\right)^{-1} = O \boxtimes \tilde{\Theta}^{-k'},$$
while for $E = O_\alpha$, we have
$$L = \det R_{p_23!}(m_{12}O_\alpha \otimes p_{13}^*P \otimes p_1^*F)^{-1} \cong \det ((-1,1)^*P \otimes p_1^*F)^{-1} \cong \mathcal{P}^{r'} \otimes \Theta^{-k'}.$$ 

The calculation for $E = \Theta$ is more involved. We show
$$L = \det R_{p_23!}(m^*\Theta \otimes p_{13}^*P \otimes p_1^*F)^{-1} = \left(\Theta^{-\chi'-k'} \boxtimes \tilde{\Theta}^{-k'-r'}\right) \otimes \mathcal{P}^{r'+k'}.$$

Observe that
$$m^*\Theta = (1 \times \Phi)^*P \otimes p_1^*\Theta \otimes p_2^*\Theta = (1 \times \Phi)^*(P \otimes p_2^*\tilde{\Theta}^{-1}) \otimes p_1^*\Theta.$$

We calculate
$$L^{-1} = \det R_{p_23!}((1 \times \Phi)^*(P \otimes p_2^*\tilde{\Theta}^{-1}) \otimes p_{13}^*P \otimes p_1^*(F \otimes \Theta))$$
$$= (\Phi \times 1)^* \det R_{p_{23}!}(p_{12}^*P \otimes p_{13}^*P \otimes p_1^*(F \otimes \Theta) \otimes p_2^*\tilde{\Theta}^{-1})$$
$$= (\Phi \times 1)^* \det R_{p_{23}!}((1 \times \tilde{m})^*P \otimes p_1^*(F \otimes \Theta) \otimes p_2^*\tilde{\Theta}^{-1})$$
where we noted again that over $A \times \tilde{A} \times \tilde{A}$ we have
$$p_{12}^*P \otimes p_{13}^*P = (1 \times \tilde{m})^*P.$$

Using that
$$\chi(F \otimes \Theta) = r' + \chi' + 2k',$$
we continue the calculation
$$L = (\Phi \times 1)^* \det \left(R_{p_{23}!}((1 \times \tilde{m})^*P \otimes p_1^*(F \otimes \Theta)) \otimes pr_1^*\tilde{\Theta}^{-1}\right)^{-1}$$
$$= (\Phi \times 1)^* \left((\det R_{p_{23}!}((1 \times \tilde{m})^*P \otimes p_1^*(F \otimes \Theta)) \otimes pr_1^*\tilde{\Theta}^{-r'-\chi'-2k'}\right)^{-1}$$
$$= (\Phi \times 1)^* (\tilde{m}^* \det R_{p_{23}!}(P \otimes p_1^*(F \otimes \Theta))^{-1} \otimes \Phi^*\tilde{\Theta}^{r'+\chi'+2k'})$$
$$= (\Phi \times 1)^* (\tilde{m}^* \det R_{p_{23}!}(P \otimes p_1^*(F \otimes \Theta))^{-1} \otimes \Phi^*\tilde{\Theta}^{-r'-\chi'-2k'})$$
$$= (\Phi \times 1)^* (\tilde{m}^* \tilde{\Theta}^{-r'-k'} \otimes \Theta^{-r'-\chi'-2k'})$$

Noting again that
$$\tilde{m}^* \tilde{\Theta} = (\tilde{\Phi} \times 1)^*P \otimes p_1^*\tilde{\Theta} \otimes p_2^*\tilde{\Theta},$$
we obtain the result
\[ L = (\Phi \times 1)^* P^{-r' - k'} \otimes (\Phi^* \hat{\Theta}^{-r' - k'} \boxtimes \hat{\Theta}^{-r' - k'}) \otimes \Theta^{-r' - \chi' - 2k'} \]
\[ = (-1, 1)^* P^{-r' - k'} \otimes (\Theta^{r' + k'} \boxtimes \hat{\Theta}^{-r' - k'}) \otimes \Theta^{-r' - \chi' - 2k'} \]
\[ = P^{r' + k'} \otimes (\Theta^{-\chi' - k'} \boxtimes \hat{\Theta}^{-r' - k}). \]

The lemma is now proved.

The only detail that still needs clarification is the fact that
\[ \det RS(F \otimes \Theta) = \hat{\Theta}^{r' + k}. \]

Here, we use that \( \Theta \) is symmetric so that \( \det \hat{\Theta}^k = \hat{\Theta}^k \). This statement follows for instance by taking determinants in Lemma 2(ii) in [O2]; that lemma is stated for odd numerics, but the proof carries through in general. Now, letting
\[ M_F = \det RS(F \otimes \Theta) \]
observes that \( M_F \) depends on the rank, Euler characteristic, determinant and determinant of the Fourier-Mukai transform of \( F \). If \( F_1 \) and \( F_2 \) are two such sheaves, we write
\[ f = F_1 - F_2 = O_Z - O_W, \]
where \( a(Z) = a(W) \). But then
\[ M_{F_1} \otimes M_{F_2}^{-1} = M_f = \det RS(\Theta \otimes f) = \det RS(\Theta \otimes (O_Z - O_W)) = P_{a(Z) - a(W)} = O. \]

Therefore, it suffices to assume that
\[ F = O^{r'} \oplus (k' \Theta - \Theta) \oplus O^\chi' - r'. \]

Finally, note that in this case
\[ M_F = \hat{\Theta}^{r'} \otimes \det \hat{\Theta}^{k'+1} \otimes \det \hat{\Theta}^{-1} \otimes \det \Theta \otimes O^{\chi' - r'} = \hat{\Theta}^{r' + k}. \]

\[ \square \]

2.4. Construction of the strange duality map. In this subsection, we use Theorem 1 to construct the duality map \( SD \) mentioned in the introduction; see equation (5). A similar construction was achieved in [Po] in the case of curves by packaging together all the strange duality morphisms over the Jacobian.

By assumption \( (A.2) \), we have \( c_1(v \otimes w) \cdot H > 0 \). Serre duality implies that for any two stable sheaves \( E \) and \( F \) we have
\[ H^2(E \otimes F) = 0. \]
Furthermore, the locus
\[(12) \quad \Theta = \{(E, F) : h^0(E \otimes F) \neq 0\} \hookrightarrow \mathcal{M}_v \times \mathcal{M}_w \]
has expected codimension 1. The defining equation of (12) is used to prove that:

**Lemma 2.** There exists a natural morphism
\[\text{SD} : j^*E(v, w)^\vee \to \widehat{E(w, v)}\]

Here we write
\[j : A \times \hat{A} \to A \times \hat{A}\]
for the multiplication by \((-1, -1)\). Since our polarizations are in fact assumed to be symmetric, pullback by the morphism \(j\) will not be necessary.

**Proof.** To construct \(\text{SD}\), we need a natural section of the bundle
\[j^*E(v, w) \otimes \widehat{E(w, v)} = R\alpha_!(\Theta_w \boxtimes \Theta_v \otimes \alpha^* Q)\]
where
\[\alpha = (j \circ \alpha_v) \times \alpha_w : \mathcal{M}_v \times \mathcal{M}_w \to (A \times \hat{A}) \times (A \times \hat{A}),\]
and \(Q \to (A \times \hat{A}) \times (A \times \hat{A})\) is the Poincaré bundle on the self-dual abelian variety \(A \times \hat{A}\).

For simplicity let us assume that universal sheaves \(E \to \mathcal{M}_v \times A\) and \(F \to \mathcal{M}_w \times A\) exist. We form the bundle
\[\Theta = \text{det} R_p!(E \boxtimes^L F)^{-1},\]
obtained by pushforward via the projection
\[p : \mathcal{M}_v \times A \times \mathcal{M}_w \times A \to \mathcal{M}_v \times \mathcal{M}_w.\]
We claim that
\[\Theta = \Theta_w \boxtimes \Theta_v \otimes \alpha^* Q.\]
This is precisely the see-saw principle combined with Theorem 1. Indeed, the restriction of \(\Theta\) to \(\{E\} \times \mathcal{M}_w\) equals
\[\Theta_E = \Theta_v \otimes \alpha_w^* \left( (\text{det} \otimes \Theta^{-k})^\vee \boxtimes (\text{det} R\mathcal{S}(E) \otimes \tilde{\Theta}^{-k})^\vee \right) = \Theta_v \otimes \alpha^* Q|_{\{E\} \times \mathcal{M}_w}.\]
The calculation of the restriction to \(\mathcal{M}_v \times \{F\}\) is similar:
\[\Theta_F = \Theta_w \otimes (\alpha_v)^* \left( (\text{det} F \otimes \Theta^{-k})^\vee \boxtimes (\text{det} R\mathcal{S}(F) \otimes \tilde{\Theta}^{-k})^\vee \right) = \Theta_w \otimes \alpha^* Q|_{\mathcal{M}_v \times \{F\}}.\]
We can now complete the proof. The locus where
\[\text{Tor}^1(E, F) = \text{Tor}^2(E, F) = 0\]
has complement of codimension at least 2 in the product space by Proposition 0.5 in [Y2]. Along this locus, the pushforward whose determinant gives \( \Theta \) can be represented by a two step complex

\[
0 \rightarrow A_0 \xrightarrow{\sigma} A_1 \rightarrow 0
\]

which yields a section \( \text{det} \sigma \) of

\[
\Theta = A_1 \otimes A_0^{-1} = \Theta_w \boxtimes \Theta_v \otimes \alpha^* Q
\]

vanishing precisely along the theta locus (12).

\[\square\]

Conjecture 1. The morphism

\[
\text{SD} : j^! E(v, w)^\vee \rightarrow \widehat{E(w, v)}
\]

is an isomorphism.

Remark 1. Just as in the case of curves, there is a slight asymmetry in the roles of \( v \) and \( w \) in the strange duality morphism (4): on one side, the determinant and determinant of the Fourier-Mukai vary, while on the other side these invariants are kept fixed. However, just as in the case of curves [Pol], the above reformulation makes it clear that

Corollary 1. If the duality morphism (4) is an isomorphism for the pair \( (v, w) \), then it is an isomorphism for the pair \( (w, v) \).

3. A CLASS OF SEMIHOMOGENEOUS BUNDLES RELEVANT TO STRANGE DUALITY

In this section, we define and study a new class of semihomogeneous bundles

\[
W(P) \rightarrow A \times \widehat{A}
\]

indexed by triples of rational numbers

\[
P = (u, v, h),
\]

These will be useful to the study of the Verlinde bundles in Section 4.

3.1. Admissible triples. We begin with some terminology. For a triple \( P = (u, v, h) \) of rational numbers, we define the determinant\(^6\)

\[
\text{det} P = uv + h^2.
\]

The rank \( r(P) \) is the smallest common denominator of \( u, v \) and \( \text{det} P \), so that

\[
r u, \quad rv, \quad r \text{det} P
\]

\(^6\)The results of this section do not use the assumption that the polarization is principal in an essential way. The only modification for a polarization with \( \chi(\Theta) = e \) is the definition of the determinant \( \text{det} P = uv e + h^2 \).
are integers. Note that in particular
\[(rh)^2 = r \cdot r(uv + h^2) - (ru)(rv) \in \mathbb{Z}\]
hence the rational number \(rh \in \mathbb{Z}\) as well. By the minimality of \(r\), we must have
\[(r, ru, rv, r \det P) = 1.\]
Define the Euler characteristic
\[\chi(P) = r \det P = r(uv + h^2) \in \mathbb{Z}.\]
A triple of rational numbers \(P\) is called admissible if \(r(P)\) and \(\chi(P)\) are odd integers.

For each admissible triple \(P\) we define the inverse triple
\[P^{-1} = \left(\frac{u}{uv + h^2}, \frac{v}{uv + h^2}, -\frac{h}{uv + h^2}\right).
\]
In particular
\[\det P^{-1} = \frac{1}{\det P}.\]
The rank of the the triple \(P^{-1}\) is easily seen to be
\[r(P^{-1}) = r(uv + h^2) = \chi(P),\]
while the Euler characteristic equals
\[\chi(P^{-1}) = r(uv + h^2) \det P^{-1} = r(P)\]
The triple \(P^{-1}\) is clearly admissible.

Note that for a triple \(P\) of rank \(r\), we can make sense of the line bundle
\[\mathcal{O}(rP) = \left(\Theta^{ru} \otimes \hat{\Theta}^{rv}\right) \otimes P^{rh}\]
over \(A \times \hat{A}\), and of the associated morphism
\[\Phi_{rP} : A \times \hat{A} \to A \times \hat{A}.
\]

3.2. **Semihomogeneous bundles.** For each admissible tripe \(P\) we construct the minimal semihomogeneous vector bundle \(W(P)\) over \(A \times \hat{A}\) of slope determined by \(P\):
\[\mu(W(P)) = u\Theta + v\hat{\Theta} + hP.\]

**Lemma 3.** For each admissible triple \(P\), there exists a simple semihomogeneous bundle \(W(P)\) of rank
\[\text{rank } W(P) = r(P)^2\]
and of slope \(P\). Its Euler characteristic equals
\[\chi(W(P)) = \chi(P)^2.\]
Proof. In general, Mukai proved the existence of a simple semihomogeneous bundle $W(P)$ of any given slope in Corollary 6.23 of [M2]. We only need to show that the rank of $W(P)$ equals $r^2$, for $r = \text{rank } P$. We let

$$L = (ru)\Theta + (rv)\hat{\Theta} + (rh)P = \mathcal{O}(rP)$$

and note

$$\mu(W(P)) = \frac{c_1(L)}{r}.$$  

Using Theorem 7.11 of [M2], the rank of $W(P)$ can be calculated from the cardinality of the set

$$\# \left\{ K(L) \cap A[r] \times \hat{A}[r] \right\} = u^2.$$  

In fact,

$$\text{rank } W(P) = \frac{r^4}{u}.$$  

We show that $u = r^2$. Indeed,

$$(x, y) \in K(L) \iff t_x^*\Theta^{ru} \otimes t_y^*\hat{\Theta}^{rv} \otimes t_{x,y}^*P^{rh} = \Theta^{ru} \otimes \hat{\Theta}^{rv} \otimes P^{rh}$$

$$\iff \Phi(rux) \otimes \hat{\Phi}(rvy) = P_{-x}^{-rh} \otimes y^{-rh}$$

$$\iff \Phi(rux) = -rhx, \hat{\Phi}(rvy) = -rhx.$$  

We invoke Lemma 15 of the Appendix to complete the proof. The lemma applies since

$$\left( r, ru, rv, \frac{(ru)(rv) + (rh)^2}{r} \right) = (r, ru, rv, r \det P) = 1$$

by the definition of the rank.

Finally, the numerics of any semihomogeneous bundle are determined by its rank and slope

$$\text{ch } W(P) = r^2 \exp(u\Theta + v\hat{\Theta} + hP).$$

Therefore,

$$\chi(W(P)) = r^2 \frac{1}{4!}(u\Theta + v\hat{\Theta} + hP)^4 = r^2(uv + h^2)^2 = \chi(P)^2.$$  

\[\Box\]

Proposition 1. There is a unique simple semihomogeneous symmetric vector bundle $W(P)$ of rank $r^2$ and determinant $\mathcal{O}(r^2P)$.

Proof. The proof uses the ideas of Section 2.1 of [O2]. We first adjust the determinant by suitable degree 0 line bundles, if needed, to achieve determinant $\mathcal{O}(r^2P)$. Then, to obtain symmetry, pick any possibly non-symmetric bundle $W(P)$. By Theorem 7.11 in [M2], we must have

$$(-1)^*W(P) = W(P) \otimes M$$
for some line bundle \( M \) of degree 0. In fact, \( M^{r^2} = \mathcal{O} \) by comparing determinants. Pick a line bundle \( L \) of order dividing \( r^2 \) with \( L^2 = M \). This is possible since \( r \) is odd. Then \( W(P) \otimes L \) is symmetric semihomogeneous of the correct determinant.

To show uniqueness, assume \( W_1 \) and \( W_2 \) are two symmetric bundles as above, and write

\[
W_1 = W_2 \otimes M.
\]

Then

\[
W_1 = (-1)^* W_1 \implies W_2 \otimes M = W_2 \otimes M^{-1} \implies M^2 \in \Sigma(W_2).
\]

Proposition 7.1 of [M2] shows that \( \Sigma(W_2) \) has odd order, in fact equal to \( r^4 \). Thus, the assignment

\[
M \rightarrow M^2
\]

is bijective on \( \Sigma(W_2) \). Hence \( M^2 = L^2 \) for some \( L \in \Sigma(W_2) \) which shows

\[
M = L \otimes \xi \implies W_1 = W_2 \otimes \xi
\]

for a 2-torsion line bundle \( \xi \). Comparing determinants we obtain \( \xi^{r^2} = 1 \). Since \( r \) is odd, we derive that \( \xi = 1 \). Therefore, \( M \in \Sigma(W_2) \), which gives \( W_1 = W_2 \).

\[\Box\]

**Example 2.** When \( h \in \mathbb{Z} \), we write

\[
u = \frac{b}{a}, \quad v = \frac{d}{c}, \quad \text{hence } P = \left( \frac{b}{a}, \frac{d}{c}, h \right).
\]

Assume for simplicity of notation that \((a, c)\) are odd and coprime. Then \( r = ac \), \( \chi = bd + h^2 ac \) and

\[
W(P) = W_{a,b} \boxtimes W_{c,d} ^{\dagger} \otimes P^h.
\]

Here \( W_{a,b} \) is the unique simple symmetric semihomogeneous bundle on \( A \) of slope \( \frac{bd}{a} \), which has rank \( a^2 \), see [O2]. The notation \( W_{c,d} ^{\dagger} \) denotes the analogous bundle of rank \( c^2 \) on the dual abelian variety.

### 3.3. Properties of semihomogeneous bundles.

In this subsection, we make precise some of Mukai’s results in [M2] for the semihomogeneous bundles \( W(P) \) constructed above.

**Lemma 4.** For any admissible triple \( P \), we have

\[
t^*_x W(P) \cong W(P) \otimes \Phi_{rP}(x), \quad \text{for } x \in A \times \hat{A}.
\]
Proof. This follows from Proposition 6.8 of [M2], where the result is shown in greater generality. In fact, Mukai proves
\[ t_r^* x \mathcal{W}(P) = \mathcal{W}(P) \otimes \Phi_r^p(x), \]
but the two statements are clearly equivalent. Note that in the lemma, \( \Phi_r^p(x) \) is understood as a line bundle over \( A \times \hat{A} \), but we decided not to introduce separate notation to indicate this fact. \( \square \)

Lemma 5. For each admissible triple \( P \), we have
\[ \hat{\mathcal{W}}(P) \cong \mathcal{W}(P^{-1}). \]

Proof. By Lemma 3, the semihomogeneous bundle \( \mathcal{W}(P) \) is non-degenerate, so it satisfies the index theorem (the index is found in Remark 2 below). The Fourier-Mukai transform \( \hat{\mathcal{W}}(P) \) is locally free. We calculate the slope in rational Chow
\[ \mu(\hat{\mathcal{W}}(P)) = \frac{1}{(uv + h^2)}(u\Theta + v\Theta - hP). \]
Note that in rational Chow the slope of \( \hat{\mathcal{W}}(P) \) equals the slope of the bundle \( \mathcal{W}(P^{-1}) \). The rank of \( \mathcal{W}(P^{-1}) \) equals \( \chi(P)^2 \), which agrees with that of \( \hat{\mathcal{W}}(P) \). Both bundles are simple and semihomogeneous. Therefore, by Proposition 6.17 in [M2], there exists \( \xi \) a degree 0 line bundle over \( A \times \hat{A} \) such that
\[ \hat{\mathcal{W}}(P) = \mathcal{W}(P^{-1}) \otimes \xi. \]

Further argumentation is necessary to prove \( \xi \) may be taken to be trivial. Using the symmetry of the bundles involved, we conclude
\[ (-1)^* \hat{\mathcal{W}}(P) = \hat{\mathcal{W}}(P) \implies (-1)^* \mathcal{W}(P^{-1}) \otimes \xi^{-1} = \mathcal{W}(P^{-1}) \otimes \xi \implies \xi^2 \in \Sigma(\mathcal{W}(P^{-1})). \]
Since \( \Sigma(\mathcal{W}(P^{-1})) \) is shown to have odd rank in Lemma 7, the map
\[ \Sigma(\mathcal{W}(P^{-1})) \ni \tau \mapsto \tau^2 \in \Sigma(\mathcal{W}(P^{-1})) \]
is surjective. We may write \( \xi^2 = \zeta^2 \), for some \( \tau \in \Sigma(\mathcal{W}(P^{-1})) \). Setting \( \eta = \xi \otimes \zeta^{-1} \), we have \( \eta^2 = 1 \) and furthermore
\[ (13) \quad \hat{\mathcal{W}}(P) = \mathcal{W}(P^{-1}) \otimes \eta. \]
We claim that \( \eta \) is trivial. This part of the proof is more involved, and in some sense roundabout.

By (13), we compute (ignoring the shifts, all by even integers, thus not affecting the argument)
\[ R\mathcal{S} (t_{-\eta}^* \mathcal{W}(P)) = \hat{\mathcal{W}}(P) \otimes \eta^{-1} = \mathcal{W}(P^{-1}). \]
We next pushforward by the isogeny $r : A \times \hat{A} \to A \times \hat{A}$. This yields
\[
\mathcal{R}S (r^* t^*_\eta W(P)) = r_* \left( \mathcal{R}S (t^*_\eta W(P)) \right) = r_* \left( W(P^{-1}) \right).
\]
Since $\eta$ has order 2 and $r$ is odd, it follows that $t_{-\eta} \circ r = r \circ t_{-\eta}$. The left hand side above rewrites as
\[
\mathcal{R}S (t^*_{-\eta} r^* W(P)) = r_* \left( W(P^{-1}) \right) \implies r^* \overline{W(P)} \otimes \eta^{-1} = r_* \left( W(P^{-1}) \right).
\]
Taking determinants, we obtain
\[
\det r^* \overline{W(P)} \otimes \eta^{-r^8 \chi^2} = \det r_* \left( W(P^{-1}) \right).
\]
We will show that
\[
\det r^* \overline{W(P)} = \det r_* \left( W(P^{-1}) \right).
\]
This will prove that $\eta$ is of order $r^8 \chi^2$, which is odd. Since $\eta$ is also 2-torsion, it must be trivial completing the proof. We will evaluate the two determinants in (14) explicitly.

First, we consider the pullback $r^* W(P)$ which has slope equal to
\[
\mathcal{L} = r^* \mathcal{O}(P) = \Theta^{r^2 u} \otimes \Theta^{r^2 v} \otimes P^{r^2 h}.
\]
The reason for making use of the isogeny $r$ is evident here: we need $\mathcal{L}$ to be a genuine line bundle, as opposed to a fractional one. Since $r^* W(P)$ is semihomogeneous, by the classification theory contained in Propositions 6.18 and 6.2 in [M2], we conclude that
\[
r^* W(P) = \bigoplus_j \mathcal{L} \otimes U_j \otimes \ell_j
\]
where $\ell_j \to A \times \hat{A}$ are line bundles of degree 0, while $U_j$ are unipotent bundles, i.e. bundles admitting filtrations with trivial successive quotients. Comparing ranks, we find
\[
\sum_j \text{rank } U_j = r^2.
\]
Taking determinants, we conclude
\[
r^* \mathcal{O}(r^2 P) = \mathcal{L}^{r^2} \otimes \left( \sum_j \text{rank } U_j \cdot \ell_j \right) \implies \sum_j \text{rank } U_j \cdot \ell_j = 0.
\]
We now compute
\[
r^* \overline{W(P)} = \bigoplus_j \mathcal{L} \otimes U_j \otimes \ell_j = \bigoplus_j t^*_j \mathcal{L} \otimes U_j.
\]
To evaluate the determinant of the Fourier-Mukai transforms appearing in the last expression, we may assume that the unipotent bundles $U_j$ are trivial; indeed, determinants are multiplicative in exact sequences. Thus

$$\det \left( r^* \hat{W}(P) \right) = \bigotimes_j t_j^* \left( \det \hat{L} \right)^{\text{rank } U_j} = \left( \det \hat{L} \right)^{r^2}$$

after using the two identities derived above from the rank and determinant calculation.

Next, we consider the right hand side of (14). By definition, $W(P^{-1})$ has determinant $O(\chi_2 P^{-1})$. Using Proposition 3.4 of [NR], we conclude that for $r$ odd

$$\det r_* \left( W(P^{-1}) \right) = \det r_* \left( O(\chi_2 P^{-1}) \right).$$

Note that again by [NR] we have

$$r^* r_* \left( O(\chi_2 P^{-1}) \right) = \bigoplus_\alpha t_\alpha^* O(\chi_2 P^{-1})$$

where the sum runs over all $r$-torsion points $\alpha$. Taking determinants, and using that the sum of the $r$-torsion points is trivial, we conclude

$$r^* \det r_* \left( O(\chi_2 P^{-1}) \right) = O(r^6 \chi_2 P^{-1}).$$

Thus, up to some $r$-torsion point $\beta$, we have

$$\det r_* \left( O(\chi_2 P^{-1}) \right) = O(r^6 \chi_2 P^{-1}) \otimes \beta.$$

Since both sides are symmetric, it follows that $\beta$ is 2-torsion. Since $\beta$ is also $r$-torsion and $r$ is odd, we find $\beta = 0$. Thus

$$\det r_* \left( O(\chi_2 P^{-1}) \right) = O(r^6 \chi_2 P^{-1}).$$

To conclude the proof of (14) it suffices to show that

$$\det \hat{L} = O(r^4 \chi_2 P^{-1}).$$

In fact, we will show more generally that if

$$L = \Theta^a \otimes \hat{\Theta}^b \otimes P^c$$

where $a, b, c$ are integers, then

(15) $$\det \hat{L} = \Theta^{a(ab+c^2)} \otimes \hat{\Theta}^{b(ab+c^2)} \otimes P^{-c(ab+c^2)}.$$
Equality (15) certainly holds at the level of Chern classes in rational Chow by a Grothendieck-Riemann-Roch calculation. Since both sides are symmetric, they differ by a 2-torsion point $\gamma$, which we show to be trivial. To this end, we let

$$\iota : \hat{A} \to A \times \hat{A}$$

be the inclusion of the zero section. We prove

$$\iota^* \gamma = 1.$$

A similar argument for the inclusion

$$j : A \to A \times \hat{A}$$

shows that $j^* \gamma = 1$. This implies that $\gamma$ is trivial, completing the proof.

The above claim that $\iota^* \gamma = 1$ is precisely the statement that

$$\iota^* \det \hat{L} = \hat{\Theta}^{b(ab+c^2)} \iff \det \pi_* L = \hat{\Theta}^{b(ab+c^2)},$$

where the projection $\pi : A \times \hat{A} \to A$ is the transpose of the inclusion $\iota$. Note that

$$\pi_* L = \Theta^a \otimes \pi_*(\hat{\Theta}^b \otimes P^c) = \Theta^a \otimes c^* R\hat{S}(\hat{\Theta}^b) = \Theta^a \otimes c^* W_{b,1}.$$

For the last equality, Proposition 2 of [O2] was used. To carry out the remainder of the proof, we use the same ideas as above. First, write $c^2$ in lowest terms $\frac{\beta}{\alpha}$, and let $W_{\alpha,\beta}$ be the bundle constructed in [O2] of this given slope. The bundle $c^* W_{b,1}$ is semihomogeneous of the same slope as $W_{\alpha,\beta}$, hence by the classification theory of [M2] we must have

$$c^* W_{b,1} = \bigoplus_j W_{\alpha,\beta} \otimes U_j \otimes m_j$$

for some unipotent $U_j$ and some line bundles $m_j$. Comparing ranks and determinants, we have

$$\sum_j \text{rank } U_j = (b/\alpha)^2, \quad \sum_j \text{rank } U_j \cdot m_j = 0.$$

Next,

$$\pi_* L = \Theta^a \otimes c^* W_{b,1} = \Theta^a \otimes W_{\alpha,\beta} \otimes \bigoplus_j U_j \otimes m_j = W_{\alpha,\beta+a} \otimes \bigoplus_j U_j \otimes m_j.$$

Therefore,

$$\det \pi_* L = \bigotimes_j \det R\hat{S}(W_{\alpha,\beta+a} \otimes U_j \otimes m_j) = \bigotimes_j t^*_m \det R\hat{S}(W_{\alpha,\beta+a} \otimes U_j).$$
Next, we may assume just as above that the bundle $U_j$ is trivial, since determinants are multiplicative in exact sequences, split or non-split. Thus,

$$\det \pi_* L = \bigotimes_j t_{m_j} \det R\mathcal{S}(W_{\alpha,\beta+a})^{\operatorname{rank} U_j} = \bigotimes_j t_{m_j}^* \left( \hat{\Theta}^{\alpha \cdot (\beta + a)} \right)^{\operatorname{rank} U_j}.$$

Proposition 2 of [O2] was used to compute the determinant of the Fourier-Mukai transform in the last equation. Using the relations derived from determinant and rank, we conclude

$$\det \pi_* L = (b/\alpha)^2 \cdot (\beta + \alpha \cdot a) \cdot \hat{\Theta} = b(ab + c^2) \cdot \hat{\Theta},$$

as claimed in (16).

The proof of the Lemma is now complete.

Lemma 6. We have

$$\Phi_{rP}^* W(P^{-1}) = \mathcal{O}(-r^2P) \otimes \mathbb{C}^2.$$

Proof. For each semihomogeneous bundle $W$ over an abelian variety $X$, such as $X = A \times \hat{A}$, Mukai introduced the subscheme

$$Z(W) = \{(x, y) \in X \times \hat{X} : t_{x}^* W = W \otimes y\}.$$

Then, letting $p$ denote the projection onto the first factor, he proved

$$p^* W = \bigoplus \mathcal{L}$$

for some line bundle $\mathcal{L}$ over $X$, cf. Lemma 3.6 [M2]. We apply this result to our situation. First, note that by the previous lemma

$$R\mathcal{S}(W(P^{-1}) \otimes P_{rx}) = t_{rP}^* W(P^{-1}) = t_{rP}^* W(P) = W(P) \otimes \Phi_{rP}(x) = W(P^{-1}) \otimes \Phi_{rP}(x) = R\mathcal{S}(t_{\Phi_{rP}(-x)}^* W(P^{-1})).$$

The shifts were omitted above for ease of notation. We derive

$$t_{\Phi_{rP}(-x)}^* W(P^{-1}) = W(P^{-1}) \otimes P_{rx}.$$

This gives a well-defined morphism

$$q : X \to Z = Z(W(P^{-1})), \quad x \mapsto (\Phi_{rP}(x), -rx)$$

such that

$$p \circ q = \Phi_{rP}.$$

Mukai’s result implies that

$$\Phi_{rP}^* W(P^{-1}) = \mathcal{L} \otimes \mathbb{C}^2.$$
for some line bundle $\mathcal{L}$. We claim $\mathcal{L} = \mathcal{O}(-r^2 P)$. Let

$$\mathcal{L}' = \mathcal{L} \otimes \mathcal{O}(r^2 P).$$

Clearly, $\mathcal{L}'$ is symmetric, and comparing determinants we see that

$$\mathcal{O}(-r^2 P)\chi^2 = \Phi_{r^2 P} \det W(P^{-1}) = \mathcal{L}\chi^2 \implies \mathcal{L}'\chi^2 = 0.$$

The first equality follows from Lemma 9 below. Since $\chi^2$ is odd, this implies $\mathcal{L}'$ is trivial as desired.

\[\square\]

**Remark 2.** The Lemma implies that $W(P)$ satisfies the index theorem with index $\text{index}\,(P)$ where

(i) if $\det P > 0$, then $\text{index}\,(P) = 2$;

(ii) if $\det P < 0$, $\text{index}\,(P)$ is either 0 or 4. The first case occurs when $u > 0$, while the second case occurs when $u < 0$.

Indeed, it suffices to establish the similar claims for $W(P^{-1})$. To this end, we invoke the Lemma, and the observation that the line bundle $\mathcal{O}(-r^2 P)$ satisfies the index theorem with the required indices.

**Lemma 7.** The group

$$\Sigma(W(P)) = \{ y \in A \times \hat{A} : W(P) \otimes Q_y = W(P) \}$$

has $r^4$ elements and can be identified with

$$\Sigma = \{ y \in (A \times \hat{A})[r] : \Phi_{\chi P^{-1}}(y) = 0 \}.$$ 

Here, $Q_y$ denotes the line bundle over $A \times \hat{A}$ associated to $y$.

**Proof.** The fact that the set $\Sigma$ has $r^4$ elements follows from Lemma 15 of the Appendix. We also have $\Sigma \subset \Sigma(W(P))$ by Lemma 4 and the fact that along $r$-torsion points

$$\Phi_{\chi P^{-1}} \circ \Phi_P = 0.$$

The claim follows, since by Proposition 7.1 of [M2] we know $\Sigma(W(P))$ should have $r^4$ elements.

\[\square\]

In a similar fashion, we obtain

**Lemma 8.** The group

$$K(W(P)) = \{ x \in A \times \hat{A} : t^*_x W(P) = W(P) \}$$

has $\chi^4$ elements, and equals

$$K = \{ x \in (A \times \hat{A})[\chi] : \Phi_{rP}(x) = 0 \}.$$
Example 3. Assume that

\[ P = \begin{pmatrix} b & d \\ a & c \end{pmatrix}, \]

with \((a, b) = (c, d) = 1\) odd coprime integers. For the bundle

\[ W(P) = W_{a,b} \boxtimes W_{c,d}^\dagger, \]

we have

\[ W(P^{-1}) = W_{d,c} \boxtimes W_{b,a}^\dagger, \]

which is indeed Fourier-Mukai dual to \(W(P)\) since by [O2], we have

\[ \hat{W}_{a,b} = W_{b,a}^\dagger. \]

Also,

\[ \Sigma(W(P)) = \hat{A}[a] \times A[c], \quad K(W(P)) = A[b] \otimes \hat{A}[d], \]

consistently with the lemmas above.

3.4. Pullbacks under isogenies. We consider the pullbacks of the semihomogeneous bundles \(W(P)\) under a special class of isogenies. We will use these calculations to support our main conjecture in Section 4.

Specifically, we consider a matrix

\[ M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

with integer entries, and set

\[ \rho = \rho_M : A \times \hat{A} \to A \times \hat{A}, \quad (x, y) \mapsto (ax + b\hat{\Phi}(y), c\Phi(x) + dy). \]

For future reference we note that

\[ \rho_M \circ \rho_N = \rho_{(MN)^\circ} \]

where

\[ M \to M^\circ \text{ is the transformation } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & -b \\ c & d \end{bmatrix}. \]

There are two other types of isogenies which will be considered and which are in fact particular cases of the ones above:

- for a triple of integers

\[ P = (u, v, h) := u\Theta + v\hat{\Theta} + hP, \]

we note that

\[ \Phi_P : A \times \hat{A} \to A \times \hat{A} \text{ can be expressed as } \Phi_P = \rho_{P^\circ}. \]
where
\[ \hat{P} = \begin{bmatrix} h & -v \\ u & h \end{bmatrix}. \]
Observe that
\[ \det \hat{P} = \det P, \]
where the left hand side denotes the determinant of the matrix, while the right hand side is the determinant of the triple \( P \).

- for a Mukai vector \( v = (r, k\Theta, \chi) \) we will consider the isogeny
\[ \Psi_v : A \times \hat{A} \to A \times \hat{A}, \ (x, y) \to (-\chi x + k\Phi(y), k\Phi(x) + ry) \]
which equals
\[ \Psi_v = \rho V, \text{ for } V = \begin{bmatrix} -\chi & -k \\ k & r \end{bmatrix}. \]
Note that \( \det V = d_v \).

We determine the pullbacks of the three bundles \( \{\Theta, \hat{\Theta}, P\} \) via the isogeny \( \rho_M \).

**Lemma 9.** The matrix of the transformation \( \rho^*_M \) on the lattice spanned by \( \{\Theta, \hat{\Theta}, P\} \) equals
\[ R = \begin{bmatrix} a^2 & -c^2 & 2ac \\ -b^2 & d^2 & 2bd \\ -ab & -cd & ad - bc \end{bmatrix}. \]

**Proof.** We calculate
\[ \rho^* \Theta = (ax + b\Phi(y))^* \Theta = \Theta^a \odot \hat{\Theta}^{-b^2} \odot P^{-ab} \]
via the see-saw theorem:
- the restriction to \( A \times \{y\} \) equals
  \[ a^* t_{b\Phi(y)}^* \Theta = a^* (\Theta \odot \Phi(b\Phi(y))) = \Theta^a \odot P^{-ab}_y \]
- the restriction to \( \{0\} \times \hat{A} \) equals \( (b\Phi)^* \Theta = \hat{\Theta}^{-b^2} \).

In a similar fashion, we calculate
\[ \rho^* \hat{\Theta} = \Theta^{-c^2} \odot \hat{\Theta}^{d^2} \odot P^{-cd} \]
Finally, the pullback
\[ \rho^* P = \Theta^{2ac} \odot \hat{\Theta}^{2bd} \odot P^{ad - bc} \]
is subtler to find. First consider the dual isogeny
\[ \rho^+ (x, y) = (dx - b\Phi(y), -c \Phi(x) + ay) \]
and compute
\[ \rho^+ \circ \rho = ad + bc. \]
\[ \Theta^{(ad+bc)^2} = (\rho^+ \circ \rho)^* \Theta = \rho^* (\rho^+)^* \Theta = \rho^* \left( \Theta^{d^2} \boxtimes \hat{\Theta}^{b^2} \otimes \mathcal{P}^{bd} \right) \]

\[ = \left( \Theta^{a^2} \boxtimes \hat{\Theta}^{c^2} \otimes \mathcal{P}^{-ab} \right)^{d^2} \otimes \left( \Theta^{-c^2} \boxtimes \hat{\Theta}^{d^2} \otimes \mathcal{P}^{-cd} \right)^{-b^2} \otimes \rho^* \mathcal{P}^{bd}. \]

This establishes that
\[ \rho^* \mathcal{P} = \Theta^{2ac} \boxtimes \hat{\Theta}^{2bd} \otimes \mathcal{P}^{ad-bc}, \]
at least up to \( bd \) torsion points. The torsion ambiguity is eliminated by restricting to \( A \times \{0\} \) and \( \{0\} \times \hat{A} \). This uses that the pullbacks of the Poincaré bundle to \( A \) and \( \hat{A} \) are:

\[ (a, c \Phi)^* \mathcal{P} = \Theta^{2ac}, \quad (b \hat{\Phi}, d)^* \mathcal{P} = \hat{\Theta}^{2bd}. \]

\[ \square \]

For future reference we note that
\[ R^{-1} = \frac{1}{(ad + bc)^2} \begin{bmatrix} d^2 & -c^2 & -2cd \\ -b^2 & a^2 & -2ab \\ bd & ac & ad - bc \end{bmatrix}. \]

4. Applications to strange duality

The theory of semihomogeneous bundles introduced in the previous section will now be applied in the setting of strange duality. In particular, we conjecture an exact formula for the Verlinde bundles.

4.1. Étale pullbacks in the setting of strange duality. Recall the conventions of Section 2. The vectors
\[ v = (r, k \Theta, \chi), \quad w = (r', k' \Theta, \chi') \]
are primitive and orthogonal i.e.,
\[ r \chi + r' \chi + 2kk' = 0. \]

The dimensions
\[ d_v = \frac{1}{2} \langle v, v \rangle = k^2 - r \chi, \quad d_w = \frac{1}{2} \langle w, w \rangle = k'^2 - r' \chi' \]
are assumed to be odd positive integers by \((A.2)\).

We continue to use the étale diagram (6). We observed that for the isogeny
\[ \Psi_v(x, y) = (-\chi x + k \hat{\Phi}(y), k \Phi(x) + r y), \]
we have
\[ \Psi^*_v \Theta_w = \Theta_w \boxtimes \mathcal{L} \implies \Psi^*_v \mathcal{E}(v, w) = H^0(K_v, \Theta_w) \otimes \mathcal{L} \]
where
\[ \mathcal{L} = \left( \Theta^{-c'k'-c} \boxtimes \hat{\Theta}^{-r'k'-r'} \right) \otimes \mathcal{P}^{r' \chi + kk'}. \]
Lemma 9 gives the matrix corresponding to the pullback via $\Psi_v$. Its inverse equals
\[
R^{-1} = \frac{1}{(r\chi - k^2)^2} \begin{bmatrix}
  r^2 & -k^2 & -2kr \\
  -k^2 & \chi^2 & 2\chi k \\
  kr & -k\chi & -r\chi - k^2
\end{bmatrix}.
\]
Therefore, the slope of the Verlinde bundles $E(v, w)$ has coordinates
\[
\frac{1}{(r\chi - k^2)^2} \begin{bmatrix}
  r^2 & -k^2 & -2kr \\
  -k^2 & \chi^2 & 2\chi k \\
  kr & -k\chi & -r\chi - k^2
\end{bmatrix} \begin{bmatrix}
  -\chi k' - \chi' k \\
  -r\chi' - r'k \\
  r'\chi + kk'
\end{bmatrix} = \frac{1}{k^2 - r\chi} \begin{bmatrix}
  r k' + r' k \\
  \chi k' + \chi' k \\
  r'\chi + kk'
\end{bmatrix}
\]
in the basis $\{\Theta, \hat{\Theta}, P\}$. The orthogonality equation (17) was used to simplify the answer.

Let us write
\[
P(v, w) = \frac{1}{d_v} \begin{bmatrix}
  r k' + r' k \\
  \chi k' + \chi' k \\
  r'\chi + kk'
\end{bmatrix}.
\]
Note that the first two entries are obtained from $c_1(v \otimes w)$ and $c_1(\hat{v} \otimes \hat{w})$, while the last entry $r'\chi + kk'$ squares to
\[
-d_v d_w - \frac{1}{2} c_1(v \otimes w)c_1(\hat{v} \otimes \hat{w}).
\]
Similarly, $P(w, v)$ is defined by reversing the roles of $v$ and $w$
\[
P(w, v) = \frac{1}{d_w} \begin{bmatrix}
  r k' + r' k \\
  \chi k' + \chi' k \\
  r'\chi + kk'
\end{bmatrix}.
\]
Thus, we obtained
\[
\mu(E(v, w)) = P(v, w), \quad \mu(E(w, v)) = P(w, v).
\]
The simplest candidates for bundles with these slopes are $W(P(v, w))$ and $W(P(w, v))$. We will check shortly in Lemma 10 that $W(P(v, w))$ does in fact pullback to a direct sum of copies of the line bundle $L$ of (18) under the morphism $\Psi_v$.

Let us however first discuss the numerics of the triples $P(v, w)$ and $P(w, v)$. A simple calculation shows that
\[
det P(v, w) = -\frac{d_w}{d_v}.
\]
Let
\[
\Delta = (r k' + r' k, \chi k' + \chi' k, d_v, d_w)
\]
When $(k, k') = 1$, we have the even simpler expression
\[
\Delta = (d_v, d_w).
\]
Clearly, $P(v, w)$ has
\[
\text{rank } P(v, w) = \frac{d_v}{\Delta}, \quad \chi(P(v, w)) = -\frac{d_w}{\Delta}.
\]
Similarly,
\[ \text{rank } \mathcal{P}(w,v) = \frac{d_w}{\Delta}, \quad \chi(\mathcal{P}(w,v)) = -\frac{d_v}{\Delta}. \]
In fact, a simple calculation shows
\[ \mathcal{P}(v,w) = -\mathcal{P}(w,v)^{-1}. \]
Thus, by Lemma 5, the bundles \( \mathcal{W}(\mathcal{P}(v,w)) \) and \( \mathcal{W}(\mathcal{P}(w,v))^\vee \) are connected by Fourier-Mukai transform:
\[ \mathcal{W}(\mathcal{P}(v,w)) \cong \mathcal{W}(\mathcal{P}(w,v))^\vee. \]

To summarize, we provided evidence for the claim that the building blocks of the Verlinde bundles \( \mathcal{E}(v,w) \) are the semihomogeneous bundles \( \mathcal{W}(\mathcal{P}(v,w)) \) with
\[ \text{rank } = \frac{d_v^2}{\Delta^2}, \quad \chi = \frac{d_w^2}{\Delta^2}, \]
consistently with the Fourier-Mukai symmetry of Section 2.4. This statement is however only true up to torsion; see Conjecture 2 below.

**Example 4.** We consider the case when \( k = 0 \) i.e. \( c_1(v) = 0 \). Then, the slope of \( \mathcal{E}(v,w) \) equals
\[ \mathcal{P}(v,w) = \begin{bmatrix} -k' \\ -k' \\ -r' \\ -r' \end{bmatrix} = \begin{bmatrix} -k' \\ -k' \\ -r' \\ -h \end{bmatrix}, \]
where we wrote \( r' = rh, \chi' = -\chi h \). Furthermore, \( \Delta = ab \), where
\[ a = (k', \chi), \quad b = (k', r). \]
Hence,
\[ \mathcal{W}(\mathcal{P}(v,w)) = \mathcal{W}_{-\frac{k'}{a}} \otimes \mathcal{W}^\dagger_{-\frac{k'}{b}} \otimes \mathcal{P}^{-h}. \]
\[ \square \]

We now check that the bundles \( \mathcal{W}(\mathcal{P}(v,w)) \) split as direct sum of line bundles under the isogeny \( \Psi_v \), as claimed above.

**Lemma 10.** The pullback
\[ \Psi_v^* \mathcal{W}(\mathcal{P}(v,w)) = \mathcal{L} \otimes \mathcal{C}^{(d_v/\Delta)^2} \]
splits as direct sum of line bundles, where
\[ \mathcal{L} = \left( \Theta^{-\chi'k-k' \chi'} \otimes \hat{\Theta}^{-r'k' - r'k} \right) \otimes \mathcal{P}^{r' \chi + kk'}. \]
Proof. We consider the dual vector \( w^\vee = (r', -k'\Theta, \chi') \) and note that for 
\[
J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]
we have 
\[
JW^\vee V^{-1} = \hat{P}(v, w).
\]
In particular
\[
\Phi_{\frac{d}{\Delta}P(v, w)} \circ \Psi_v = \rho_{\frac{d}{\Delta}P(v, w)} \circ \rho_{W^\vee} = \rho_{J^o} \circ \rho_{(W^\vee)^o} \circ \frac{d_v}{\Delta} = \rho_{J^o} \circ \Psi_{w^\vee} \circ \frac{d_v}{\Delta}.
\]

It suffices to check that \( \Psi_v^* W(P(v, w)) \) splits as direct sum of the same line bundle \( L \), since then the line bundle is identified uniquely just as in Lemma 6 above, using invariance under \((-1)\) and the determinant calculation. In particular, it suffices to check that 
\[
q : X \to \mathcal{Z}(W(P(v, w))), \ x \mapsto (\Psi_v(x), \rho_{J^o} \circ \Psi_{w^\vee}(x))
\]
is a well defined morphism, since then 
\[
p \circ q = \Psi_v
\]
and we can invoke Mukai’s result which shows that the pullback under \( p \) already splits.

It remains to prove
\[
(19) \quad t_{\Psi_v(x)}^* W(P(v, w)) = W(P(v, w)) \otimes \rho_{J^o} \Psi_{w^\vee}(x).
\]

Our general study of the semihomogeneous bundles \( W(P) \), in particular Lemma 4 shows that
\[
(20) \quad t_{\frac{d}{\Delta}y}^* W(P(v, w)) = W(P(v, w)) \otimes \Phi_{\frac{d}{\Delta}P(v, w)}(y).
\]

Fix \( x \) and write 
\[
x = \frac{d_v}{\Delta} x', \text{ for } x' \in X.
\]
Set \( y = \Psi_v(x') \). We calculate,
\[
\Phi_{\frac{d}{\Delta}P(v, w)}(y) = \Phi_{\frac{d}{\Delta}P(v, w)} \Psi_v(x') = \rho_{J^o} \Psi_{w^\vee} \frac{d_v}{\Delta}(x') = \rho_{J^o} \Psi_{w^\vee}(x).
\]
Now (19) follows from (20) by substitution.

\[\square\]
4.2. Conjectural description of the Verlinde bundles. In this subsection, we state our main conjecture expressing the Verlinde bundles in terms of the semihomogeneous vector bundles $W(P)$.

In order to make the precise statement, we need to introduce the group $Q_{v,w}$. Recall the morphism

$$
\Psi_v(x, y) = (-\chi x + k\Phi(y), k\Phi(x) + ry).
$$

We pullback line bundles over $A \times \hat{A}$ via $\Psi_v$. By Lemma 10 we have

$$
\Sigma(W(P(v, w))) \subset \text{Ker } \Psi^*_v.
$$

The quotient

$$
Q_{v,w} = \text{Ker } \Psi^*_v/\Sigma(W(P(v, w))
$$

has \( \frac{d^4}{(d_v/\Delta)^4} = \Delta^4 \) elements.

**Lemma 11.** All elements of $Q_{v,w}$ have orders dividing $\Delta$.

**Proof.** We will show in fact that the group $Q_{v,w}$ is isomorphic to $A[\Delta]$. Let $m$ and $n$ be coprime integers such that

$$
\chi m \equiv -kn \mod d_v, \quad rn \equiv -km \mod d_v.
$$

These exist by Lemma 14 of the Appendix. Then

$$
\xi^m \otimes \hat{\Phi}^*\xi^n \in \text{Ker } \Psi^*_v,
$$

for each $\xi \to A$ which is $d_v$-torsion. Indeed, we calculate

$$
\Psi^*_v \left( \xi^m \otimes \hat{\Phi}^*\xi^n \right) = (-\chi x + k\Phi(y), k\Phi(x) + ry)^* \left( \xi^m \otimes \hat{\Phi}^*\xi^n \right)
$$

$$
= \xi^{-m\chi-nk} \otimes \hat{\Phi}^*\xi^{mk+nr} = 0.
$$

If $\xi$ is in fact $d_v/\Delta$-torsion, the bundles above are in fact in $\Sigma(W(P(v, w))$ by Lemma 7 and the fact that

$$
\Phi_{d_{w,P(v,w)}}(\hat{\Phi}(\xi^n), \xi^m) = 0.
$$

The map

$$
Q_{v,w} \to A[\Delta], \quad \xi^m \otimes \hat{\Phi}(\xi^n) \mapsto \frac{d_v}{\Delta} \xi
$$

is then a group isomorphism. \(\square\)

**Conjecture 2.** Assuming (A.1) – (A.2), we have

$$
E(v, w) = \bigoplus_{\xi} W(P(v, w)) \otimes \xi^{\pm m}\xi.
$$
where the line bundles $\zeta$ belong to $Q_{v,w}$. The multiplicity of a line bundle $\zeta$ of order $\omega$ is

$$m_\omega = \frac{1}{d_v + d_w} \sum_{\delta \mid \Delta} \frac{\delta^4}{\Delta^2} \left\{ \frac{\Delta/\omega}{\delta} \right\} \left( \frac{d_v/\delta + d_w/\delta}{d_v/\delta} \right).$$

**Example 5.** When $(d_v, d_w) = 1$, we obtain $\Delta = 1$, hence we expect

$$E(v, w) = \bigoplus W(P(v, w)),$$

where the number of summands equals $\frac{1}{d_v + d_w} (d_v + d_w)$. 

**Example 6.** We consider the case $k = 0$. We have $\Psi_v = (-\chi, r)$ so

$$\text{Ker } \Psi_v^* = \hat{A}[-\chi] \times A[r].$$

Writing

$$(k', \chi) = a, \quad (k', r) = b,$$

we noted in Example 4 that

$$W(P(v, w)) = W_{-\frac{\chi}{a} \frac{r}{b}} \otimes W_{-\frac{\chi}{a} \frac{r}{b}}^\dagger \otimes \mathcal{P}^{-h}.$$ 

Then it is clear that

$$\Sigma(W(P)) = \hat{A} \left[ -\frac{\chi}{a} \right] \times A \left[ \frac{r}{b} \right].$$

Therefore

$$Q_{v,w} \cong \hat{A}[a] \times A[b].$$

There are precisely $(ab)^4$ possible torsion points in $Q_{v,w}$. For each element $\zeta \in \hat{A}[a] \times A[b]$, any choice of $\ell \in \hat{A} \times A$ such that

$$\left( \frac{\chi}{a}, \frac{r}{b} \right) \ell = \zeta$$

gives a well-defined element of $Q_{v,w}$. The conjecture claims that

$$E(v, w) = \bigoplus_\zeta W(P(v, w)) \otimes \ell^{m_\zeta}.$$ 

The line bundle $\zeta$ of order $\omega$ dividing $(a, b)$ should appear with multiplicity

$$m_\omega = \frac{1}{d_v + d_w} \sum_{\delta \mid ab} \frac{\delta^4}{(ab)^2} \left\{ \frac{ab/\omega}{\delta} \right\} \left( \frac{d_v/\delta + d_w/\delta}{d_v/\delta} \right).$$

The symmetry of the multiplicities under exchange of $v$ and $w$ in the expression above is obvious. This has the following consequence:

**Lemma 12.** Conjecture 2 implies that

$$E(v, w)^\vee \cong E(w, v).$$
Proof. Fix \( \zeta \in \mathbb{Q}_{v,w} \) of order dividing \( \omega \). We consider a summand \( W(P(v,w)) \otimes \zeta \) appearing in the conjectured expression of the Verlinde bundle \( E(v,w) \). We calculate the Fourier-Mukai dual
\[
R_S(W(P(v,w)) \otimes \zeta) = t_\zeta^* W(P(v,w)) = t_\zeta^* W(P(w,v))^\vee.
\]
Writing
\[
\zeta = \frac{d_w}{\Delta}(\zeta'),
\]
the above expression becomes
\[
t_\zeta^* W(P(w,v))^\vee = \left( W(P(w,v)) \otimes \Phi_{\frac{d_w}{\Delta}P(w,v)}(\zeta') \right)^\vee,
\]
via equation (20). We observe that \( \Phi_{\frac{d_w}{\Delta}P(w,v)}(\zeta') \) is an element of \( \mathbb{Q}_{w,v} \). Indeed, an easy calculation gives
\[
\Psi_w^* \Phi_{\frac{d_w}{\Delta}P(w,v)}(\zeta') = \Psi_v^* \left( \frac{d_w}{\Delta} \zeta' \right) = \Psi_v^* \zeta = \mathcal{O}.
\]
Note that since \( \frac{d_w}{\Delta}(\omega \zeta') = 0 \), by Lemma 4 we have
\[
\Phi_{\frac{d_w}{\Delta}P(w,v)}(\omega \zeta') \in \Sigma(W(P(w,v))).
\]
Hence \( \Phi_{\frac{d_w}{\Delta}P(w,v)}(\zeta') \) has order dividing \( \omega \) in \( \mathbb{Q}_{w,v} \). Thus, every summand appearing in \( E(v,w) \) gives rise to a corresponding summand in \( E(w,v)^\vee \) and the multiplicities match as well. \( \square \)

4.3. Invariance under twists. In this subsection we provide a simple check of Conjecture 2. In our context, various moduli spaces of sheaves can be related by simple isomorphisms. For instance, we may twist by arbitrary line bundles to change the determinant:
\[
i : \mathcal{M}_v \rightarrow \mathcal{M}_{v_0}, \quad E \mapsto E \otimes \Theta^{-\ell}
\]
where
\[
v_0 = v \exp(-\ell \Theta).
\]
Clearly, for the Mukai vector
\[
w_0 = w \exp(\ell \Theta)
\]
we obtain
\[
i^* \Theta_{w_0} = \Theta_w.
\]
We show that

Lemma 13. If Conjecture 2 is true for the pair \((v, w)\), then it is true for the pair \((v_0, w_0)\).
Proof. We let \( \alpha_0 : \mathfrak{M}_{v_0} \to A \times \hat{A}, \ E \mapsto (\det R_S(E) \otimes \hat{\Theta}^{-r \ell - k}, \det E \otimes \Theta^{r \ell - k}) \) be the Albanese map. We show
\[
\alpha_0 \circ i = \rho_M \circ \alpha
\] (21)
where \( M \) is the matrix
\[
M = \begin{bmatrix} 1 & -\ell \\ 0 & 1 \end{bmatrix},
\]
so that \( \rho_M(x, y) = (x - \ell \hat{\Phi}(y), y) \).

Indeed, we only need to prove that
\[
\det R_S(E \otimes \Theta^{-\ell}) = (\det R_S(E) \otimes \hat{\Theta}^{-r \ell}) \otimes P_{-\ell \hat{\Phi}(\det E \otimes \Theta^{-k})}.
\]
This identity will be established by the methods of Section 2. First, for a sheaf \( E \) of rank \( r \) and first Chern class \( k \) \( \Theta \), set\[
f(E) = \det R_S(E \otimes \Theta^{-\ell})^\vee \otimes (\det R_S(E) \otimes \hat{\Theta}^{-r \ell}) \otimes P_{-\ell \hat{\Phi}(\det E \otimes \Theta^{-k})}.
\]
We show \( f(E) = \mathcal{O} \). As usual, \( f \) depends only on the \( K \)-theory class of \( E \). We may assume that \( E = \mathcal{O}_y \) or \( E \) is a line bundle. Note that\[
f(\mathcal{O}_y) = \mathcal{O}, \ f(\Theta^k) = \mathcal{O}.
\]
For an arbitrary line bundle \( E \), the proof follows. Indeed, it suffices to observe that for any \( y \) of degree 0 we have
\[
f(E \otimes y) = \det R_S(E \otimes y \otimes \Theta^{-\ell})^\vee \otimes (\det R_S(E \otimes y) \otimes \hat{\Theta}^{-r \ell}) \otimes P_{-\ell \hat{\Phi}(\det E \otimes y \otimes \Theta^{-k})}
\]
\[
= t_y^* \det \left( R_S(E \otimes \Theta^{-\ell}) \right)^\vee \otimes (t_y^* \det R_S(E) \otimes \hat{\Theta}^{-r \ell}) \otimes P_{-\ell \hat{\Phi}(\det E \otimes \Theta^{-k}) \otimes P_{-r \ell \hat{\Phi}(y)}}
\]
\[
= t_y^* f(E).
\]
With (21) understood, note that \( \rho_M^* \mathcal{E}(v_0, w_0) = \mathcal{E}(v, w) \).

We confirm the expression of Conjecture 2 obeys the same equality. It suffices to show that
\[
(i) \ \rho_M^* \mathcal{W}(\mathcal{P}(v_0, w_0)) = \mathcal{W}(\mathcal{P}(v, w))
\]
\[
(ii) \ \rho_M^* : \mathcal{Q}_{v_0, w_0} \to \mathcal{Q}_{v, w} \text{ is an isomorphism}.
\]
These two statements are equivalent to the assertions
\[
(i) \ \rho_M^* \mathcal{P}(v_0, w_0) = \mathcal{P}(v, w)
\]
(ii) $\Psi v_0 \circ \rho_L = \rho_M \circ \Psi v$ for the matrix $L = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix}$,
which are immediate verifications. For instance, (i) uses the matrix of the pullback $\rho_M^*$
which was calculated to be $\begin{bmatrix} 1 & 0 & 0 \\ -\ell^2 & 1 & -2\ell \\ \ell & 0 & 1 \end{bmatrix}$.

5. The Verlinde bundles in degree 0

This section contains the proof of the main results. In particular, we confirm Conjecture 2 in degree 0 in Theorem 3. The main step is the Lefschetz-Riemann-Roch trace calculation of Theorem 1. We also prove the Fourier-Mukai symmetries of the Verlinde bundles in Theorem 3.

5.1. Setup. We specialize to the case $c_1(v) = 0$ i.e. we assume

$$v = (r, 0, \chi), \ w = (rh, k\Theta, -\chi h)$$

with $r, \chi$ odd, and $(r, \chi) = 1$. The integers $(rh, k, -\chi h)$ were previously denoted $(r', k', \chi')$, but the new notation should make the exposition easier to follow.

The usual étale diagram

$$
\begin{array}{ccc}
K_v \times A \times \hat{A} & \longrightarrow & M_v \\
p & & \alpha \\
A \times \hat{A} & \longrightarrow & A \times \hat{A} \end{array}
$$

takes a simpler form. In particular,

$$\Psi(x, y) = (-\chi x, ry).$$

In Subsection 2.3, we calculated the pullback

(22) $\tau^* \Theta_w = \Theta_w \boxtimes \left( \Theta^{-\chi} \boxtimes \hat{\Theta}^r \right)^k \otimes \left( \mathcal{P}^{r\chi} \right)^h.$

Consequently, we have

(23) $\Psi^* \mathcal{E}(v, w) = p_\ast \tau^* \Theta_w = H^0(K_v, \Theta_w) \otimes \left( \Theta^{-\chi} \boxtimes \hat{\Theta}^r \right)^k \otimes \Psi^* \mathcal{P}^{-h}.$

5.2. Group actions. We consider actions of the torsion group

$$G = A[-\chi] \times \hat{A}[r]$$
on the three spaces $K_v, A \times \hat{A}$ and $M_v$ appearing in the diagram. The action of $G$ on $A \times \hat{A}$ is given by translation on both factors, the action on $M_v$ is trivial, while the action on $K_v$ is given by

$$(x, y) \cdot E = t^*_{-x} E \otimes y^{-1}.$$
There are induced actions of a subgroup $K$ of $G$ on the theta bundles, which we now describe. Writing

$$a = \gcd(k, \chi), \quad b = \gcd(k, r),$$

we conclude that

$$\gcd(a, b) = 1 \text{ and } a, b \text{ are odd.}$$

We set

$$K = A[a] \times \widehat{A}[b] \hookrightarrow G.$$ 

The morphisms $\tau$ and $\Psi$ are invariant under the action of $G$, hence also under the action of $K$. The bundles $\tau^*\Theta_w$ and $\Psi^*P$ are naturally $K$-equivariant. We claim that

$$\Theta_w \to K_v$$

is $K$-equivariant as well.

Certainly, $\Theta^{-\chi_k} \boxtimes \widehat{\Theta}^{-rk}$ carries a linearization of the Heisenberg group

$$H = H[-\chi_k] \times \widehat{H}[rk]$$

where

$$0 \to \mathbb{C}^* \times \mathbb{C}^* \to H \to A[-\chi_k] \times \widehat{A}[rk] \to 0.$$ 

Our convention is that $H[m]$ denotes the Heisenberg group of $\Theta^m$ consisting of pairs $(x, f)$ where

$$f : t^*\Theta^m \to \Theta^m$$

is an isomorphism, while $\widehat{H}[n]$ denotes the Heisenberg group of $\widehat{\Theta}^n$. We have a natural morphism

$$\iota : H[a] \times \widehat{H}[b] \to H$$

which over the centers restricts to

$$(\alpha, \beta) \to (\alpha^{-\chi_k}, \beta^{\chi_k}).$$

It is now useful to pass to the finite Heisenberg groups $\bar{H}$ obtained by restricting the centers to roots of unity. For instance

$$0 \to \mu_{-\chi_k} \times \mu_{rk} \to \bar{H} \to A[-\chi_k] \times \widehat{A}[rk] \to 0,$$

and similarly

$$0 \to \mu_a \to \bar{H}[a] \to A[a] \to 0, \quad 0 \to \mu_b \to \bar{H}[b] \to \widehat{A}[b] \to 0.$$ 

Since the center of $H[a] \times H[b]$ is trivial under $\iota$, we obtain a morphism

$$j : A[a] \times \widehat{A}[b] \to \bar{H}.$$
Furthermore, since \( a, b \) are odd, the identification

\[
\Theta^{-xk} \boxtimes \hat{\Theta}^{-y} \cong (a, b)^* \left( \Theta^{-xk} \boxtimes \hat{\Theta}^{-y} \right)
\]

is compatible with the action of the group

\[
K = A[a] \times \hat{A}[b]
\]

coming from \( j \) on the left, and via the pullback on the right. Using (22), and the natural \( K \)-action on line bundles \( \tau^* \Theta_w \) and \( \Psi^* \mathcal{P} \), we obtain a \( K \)-linearization of

\[
\Theta_w \to K_v.
\]

5.3. Trace calculations. As a result of the above discussion, the space of generalized theta functions

\[
H^0(K_v, \Theta_w)
\]

carries a \( K \)-action as well. We will determine the \( K \)-representation \( H^0(K_v, \Theta_w) \) explicitly. The assumptions are \((A.1)\) and in lieu of \((A.2)\) we make the weaker

\[
(A.2)' \quad \Theta_w \text{ belongs to the movable cone of } K_v; \text{ see equation (2) for numerical conditions which ensures this.}
\]

**Theorem 2.** Consider \( \zeta = (x, y) \in A[a] \times \hat{A}[b] \) of order \( \delta \). Then the trace of \( \zeta \) on the space of generalized theta functions equals

\[
\text{Trace} (\zeta, H^0(K_v, \Theta_w)) = \frac{d_v^2}{d_v + d_w} \left( \frac{d_v/\delta + d_w/\delta}{d_v/\delta} \right).
\]

**Proof.** The proof is an application of the Lefschetz-Riemann-Roch theorem, as in [O2]. The details, both quantitative and qualitative, are however very different.

**Step 1.** In order to facilitate the application of the Lefschetz-Riemann-Roch, we begin by transferring the calculation to a smooth moduli space over which the Theta line bundle carries no higher cohomology. This is a consequence of the fact that

\[
\Theta_w \to K_v
\]

belongs to the movable cone. Furthermore, the smooth birational models of \( K_v \) are obtained as moduli spaces of Bridgeland stable objects \( K_v(\sigma) \) for suitable stability conditions \( \sigma \). The birational isomorphism extends to codimension 1.

Even though not strictly necessary for the argument, we describe briefly the moduli space \( K_v(\sigma) \), for the benefit of the reader. First, the stability condition takes the form

\[
\sigma = (Z_{\beta, \omega}, A_{\beta, \omega})
\]
for suitable divisor classes \( \beta \) and \( \omega \). Here, the central charge is given by

\[
Z_{\beta,\omega}(E^\bullet) = \langle \exp(\beta + i\omega), \text{ch}(E^\bullet) \rangle.
\]

The central charge determines the phase function

\[
\phi_{\beta,\omega}(E^\bullet) = \frac{1}{\pi} \text{Arg} \ Z_{\beta,\omega}(E^\bullet).
\]

Next, we recall the description of the heart \( A_{\beta,\omega} \). Objects \( E^\bullet \in A_{\beta,\omega} \) are two step complexes such that

\[
H^{-1}(E^\bullet) \in F_{\beta,\omega} \quad \text{and} \quad H^0(E^\bullet) \in T_{\beta,\omega}.
\]

The torsion pair \((F_{\beta,\omega}, T_{\beta,\omega})\) used above is defined in terms of the Harder-Narasimhan filtration with respect to the usual slope function \( \mu_\omega \) given by the ample class \( \omega \). Specifically,

- the sheaves in \( F_{\beta,\omega} \) have the property that all factors of the \( \mu_\omega \)-Harder-Narasimhan filtration have slope \( \mu_\omega \leq \beta \cdot \omega \);
- by contrast, the sheaves in \( T_{\beta,\omega} \) have the property that all factors of the \( \mu_\omega \)-Harder-Narasimhan filtration have slope \( \mu_\omega > \beta \cdot \omega \) (or are torsion).

To form the moduli space \( K_v(\sigma) \), we consider objects in the abelian category \( A_{\beta,\omega} \) which are semi-stable with respect to the phase function \( \phi_{\beta,\omega} \) above.

As before, the moduli space \( K_v(\sigma) \) carries an action of \( \zeta \in A[a] \times \tilde{A}[b] \):

\[
\zeta \cdot E^\bullet = t^*_xE^\bullet \otimes y^{-1}.
\]

The action similarly lifts to the theta bundle. As a result of our discussion, we obtain a \( \zeta \)-equivariant identification

\[
H^0(K_v, \Theta_w) = H^0(K_v(\sigma), \Theta_w).
\]

To prove the theorem, it suffices to show

\[
\text{Trace } (\zeta, H^0(K_v(\sigma), \Theta_w)) = \frac{d_v^2}{d_v + d_w} \left( \frac{d_v/\delta + d_w/\delta}{d_v/\delta} \right).
\]

Step 2. To this end, we first find the fixed points of the action of \( \zeta \):

\[
t^*_xE^\bullet \cong E^\bullet \otimes y^{-1}, \quad E^\bullet \in K_v(\sigma).
\]

Possibly replacing \( a \) and \( b \) by some of their divisors, we may assume that the order of \( x \) is \( a \) and the order of \( y \) is \( b \). Of course

\[
a \mid \chi, \ b \mid r \implies \gcd(a, b) = 1.
\]

We have \( \delta = ab \).
Now,
\[
t^*_x E^\bullet \cong E^\bullet \otimes y^{-1} \implies t^*_a E^\bullet \cong E^\bullet \otimes y^a \implies E^\bullet \cong E^\bullet \otimes y^a.
\]
Since the order of $y$ is coprime to $a$, we obtain
\[
E^\bullet \cong E^\bullet \otimes y \text{ and consequently } t^*_x E^\bullet \cong E^\bullet.
\]

Consider the abelian cover
\[
p : A \to A'' = A/\langle x \rangle.
\]
The Galois group $G$ of the cover is generated by translations by $x$. Let $\hat{G}$ be the dual group, and pick a generator $x''$ of $\hat{G}$. This corresponds to a line bundle $x'' \to A''$ of order $a$ which determines the cover $p$. Then
\[
E^\bullet = p^* E''
\]
for some complex $E''$ on $A''$ with
\[
\text{rank } E'' = r, \quad \chi(E'') = \frac{\chi}{a}.
\]
We have
\[
p^* E'' = E^\bullet \cong E^\bullet \otimes y = p^*(E'' \otimes y'')
\]
where
\[
y = p^* y''.
\]
Therefore,
\[
E'' = E'' \otimes y'' \otimes x^{n\ell}
\]
for some $\ell$. Replacing $y''$ by $y'' \otimes x^{n\ell}$ we may assume
\[
E'' = E'' \otimes y''.
\]
Note that
\[
y^b = p^* y'^b = 1 \implies y'^b \in \langle x'' \rangle \implies y'^{ab} = 1.
\]
Therefore, the order $\Delta$ of $y''$ satisfies $\Delta|ab$. We will show shortly that in fact $\Delta = b$.

Now, let
\[
\pi : A' \to A''
\]
be the cover determined by $y''$ which has degree $\Delta$. Let $G'$ denote the Galois group of the cover, and let $\hat{G}'$ be the dual group which is generated by $y''$. We collect the following facts about complexes over arbitrary abelian covers:

(i) if $E'' \cong E'' \otimes y''$, then $E''$ is the pushforward of a complex $E' \to A'$:
\[
E'' = \pi_* E'
\]

(ii) $\pi_* E'_1 = \pi_* E'_2$ iff $E'_1 \cong \beta^* E'_2$ for some $\beta \in G'$
(iii) \[ \pi^* \pi_* E^* = \bigotimes_{\beta \in G'} \beta^* E^* \]

(iv) the action of \( y'' \in \hat{G}' \) on \( \pi^* \pi_* E^* \) leaves each of the factors \( \beta^* E^* \) invariant. The weight of the action on each factor equals \( \langle y'', \beta \rangle \).

Specifically, (i) is proven in [BM], while (ii), (iii) and (iv) follow by the methods of [NR].

We conclude \( E^{**} = \pi_* E^* \)

where
\[
\text{rank } E^* = \frac{r}{\Delta}, \quad \chi(E^*) = \chi(E^{**}) = \frac{\chi}{a}.
\]

In particular, \( \Delta | r \) hence \( \Delta \) is coprime to \( a \). Thus \( \Delta \) divides \( b \). On the other hand,
\[
y''^\Delta = 1 \implies y^\Delta = p^* y''^\Delta = 1 \implies b | \Delta.
\]

Hence \( \Delta = b \). Now, the preimage of \( y \) under \( p \) contains at most one element of order \( b \) since the preimage of \( 0 \) contains at most one such element. Indeed, \( p^{-1}(0) \) consists only in multiples of \( x'' \) which all have order \( a \). This fixes \( y'' \) and therefore the maps \( \pi \) and \( p \) uniquely. Furthermore
\[
E^* = p^* \pi_* E^*, \quad \text{rank } E^* = \frac{r}{b}, \quad \chi(E^*) = \frac{\chi}{a}.
\]

At this point we briefly discuss stability. Consider the stability condition over \( A' \) given by
\[
\sigma' = \pi^* \sigma'',
\]

where \( \sigma'' \) over \( A'' \) is chosen so that
\[
p^* \sigma'' = \sigma.
\]

It can be immediately checked from definitions that \( E^* \) is \( \sigma \)-stable implies that \( E^* \) is \( \sigma' \)-stable, and conversely the stability of \( E^* \) implies (semi)stability of \( E^* \). These choices of stability conditions will be assumed below.

Next, we claim that there are \((ab)^3\) fixed loci all isomorphic to \( K_{\nu'}(A') \) for
\[
\nu' = \left( \frac{r}{b}, 0, \frac{\chi}{a} \right).
\]

This accounts for determinant requirements and possible repetitions. Precisely, we have
\[
\det E^* \neq 0 \implies \det \pi_* E^* \in \langle x'' \rangle.
\]

Writing
\[
\det E^* = \pi^* M''
\]
we obtain by the proof of Lemma 3.4 in [NR] that
\[ \det \pi_*E' = \det \pi_\pi^*M'' = \det \left( \bigoplus_{j=1}^b M'' \otimes y'^j \right) = M''^b \in \langle x'' \rangle \implies M'' \in \langle x'' \rangle + \tilde{A}''[b], \]
where above we used that the order of \( x'' \) is \( a \) which is coprime to \( b \). Thus
\[ \det E' \in \pi^*(x'') + \pi^*\tilde{A}''[b]. \]
There are \( ab^3 \) choices for the determinant of \( E' \), since the torsion point \( y'' \in \tilde{A}''[b] \) pulls back trivially to \( A' \). The calculation of the Fourier-Mukai determinant is similar. We have
\[ \det \hat{E}' = \det \hat{p}_\pi \hat{E}' = O. \]
Write
\[ \hat{\pi}^* \det \hat{E}' = \hat{p}^* M \]
and observe that this gives
\[ \det \hat{p}_\pi \hat{p}^* M = M^a = O \implies M \in A[a]. \]
This gives \( a^3 \) options for \( \hat{p}^* M \) since \( x \in A[a] \) pulls back trivially. Finally, this identifies \( \det \hat{E}' \) up to the \( b \) elements in the kernel of \( \pi \):
\[ \pi(\det \hat{E}') = p(M) \in p(A[a]) \implies \det \hat{E}' \in \pi^{-1}(p(A[a])). \]
We obtain \((ab^3)(a^3b)\) fixed loci. However, this answer does not account for repetitions. We observe that
\[ p^* \pi_* E_1' = p^* \pi_* E_2' \iff \pi_* E_1' = \pi_* E_2' \otimes x'^{\alpha} \iff E_1' = \beta^* E_2' \otimes \pi^* x'^{\alpha}, \beta \in G'. \]
The second statement above follows pushing forward via \( p \) and using stability. The third statement is contained in (ii). As a result, we are left with only \((ab^3)\) fixed loci. Indeed, via the above equivalence we only obtain \( b^3 \) choices for the determinant, if we require
\[ \det E' \in \pi^* \tilde{A}''[b]. \]
Similarly, the determinant of the Fourier-Mukai dual can be fixed in \( a^3 \) ways. It is easy to see there are no other repetitions.

**Step 3.** We apply the Lefschetz-Riemann-Roch theorem to calculate the trace in the theorem by summing the contributions from the fixed loci \( K_{A'}(A') \). We obtain
\[ \text{Trace}(\zeta, H^0(K_{A'}, \Theta_w)) = (ab)^3 \int_{K_{A'}}(K_{A'}) \cdot i^* \text{ch}_Z(\Theta_w)(\zeta) \prod_{z \neq 1} (\text{ch}_{\langle \zeta^{-1}, z \rangle}N_z^\vee)^{-1}. \]
Here \( Z \) is the cyclic group generated by \( \zeta \). The notation we used is as follows:
\[ i^* \text{ch}_Z(\Theta_w) \in H^*_Z(K_{A'}(A')) = \text{Rep}(Z) \otimes H^*(K_{A'}(A')) \]
is the restriction of the equivariant Chern character via inclusion

\[ i : K_v' \to K_v, \]

followed by the evaluation against \( \zeta \in \mathbb{Z} \) in the representation ring. The normal bundle of the inclusion \( i \) splits into eigenbundles \( N_z \) indexed by elements in the character group \( z \in \hat{Z} \). Finally, we write

\[ \text{ch}_t(N) = \prod_i (1 + t e_i) \]

for any bundle \( N \) with Chern roots \( x_1, \ldots, x_{\ell} \). The prefactor \((ab)^3\) comes from the fact that all fixed loci will have identical trace contributions.

**Step 4.** We evaluate the integral above explicitly. We begin by computing the normal bundles \( N_z \) in the expression above. We will compute the eigenvalues of the action of \( \zeta \) on \( T_{E \bullet}K_v \). We first consider the similar action on \( T_{E \bullet}M_v \) and use the morphism

\[ \alpha : M_v \to A \times \hat{A} \]

to find the eigenvalues on the fiber.

The tangent space to \( \mathfrak{M}_v \) at a fixed point \( E^\bullet \) was calculated in [I]:

\[
T_{E \bullet} \mathfrak{M}_v = \bigoplus_{\alpha} \text{Ext}^1(\pi_*E^\bullet, \pi_*E^\bullet \otimes x^{\sigma}) = \bigoplus_{\alpha} \text{Ext}^1(\pi_*E^\bullet, \pi_*E^\bullet \otimes \pi_*x^{\sigma})
\]

We claim that

\[ T_{\alpha,\beta} = \text{Ext}^1(\beta^*E^\bullet, E^\bullet \otimes x^{\sigma}) \]

are the isotypical components of the tangent space. Indeed, by (iv) in Step 1, \( y \) acts on each summand via the root of unity \( \langle \beta, y \rangle \) of order \( b \). The action of \( x \) also leaves the subspace invariant, and the action has weight \( \langle x, x^{\sigma} \rangle \) which is root of unity of order \( a \). As \( \alpha, \beta \) vary, we obtain all the \((ab)\)-roots of unity as eigenvalues.

We calculate the Chern roots of the eigenbundles \( T_{\alpha,\beta} \). Clearly, by Hirzebruch-Riemann-Roch, the Chern character of the virtual bundles

\[ \sum_{i=0}^2 (-1)^i \text{Ext}^i(\beta^*E^\bullet, E^\bullet \otimes \pi^*x^{\sigma}) \]

must stay constant as \( \beta \) and \( \pi^*x^{\sigma} \) vary in the abelian variety \( \hat{A}' \). (The index \( i \) is checked to have the correct range, by stability and duality.) Furthermore, for \( \alpha, \beta \) not both trivial
we have
\[
\text{Ext}^0(\beta^*E^*, E^* \otimes \pi^*x'^\alpha) = \text{Ext}^2(\beta^*E^*, E^* \otimes \pi^*x'^\alpha) = 0
\]
while for trivial \(\alpha\) and \(\beta\) the two dimensions are 1, by stability of \(E^*\). Indeed, for nontrivial \((\alpha, \beta)\), we calculate by duality
\[
\text{Ext}^2(\beta^*E^*, E^* \otimes \pi^*x'^\alpha) = \text{Ext}^0(\beta^*E^* \otimes \pi^*x'^\alpha, E^*) = \text{Ext}^0((\beta^{-1})^*E^*, E^* \otimes \pi^*x'^{-\alpha})
\]
so it suffices to prove the statement about \(\text{Ext}^0\). This is immediate since any non-zero morphism
\[
\beta^*E^* \to E^* \otimes \pi^*x'^\alpha
\]
is an isomorphism inducing a map
\[
\pi_*\beta^*E^* = \pi_*E^* \to \pi_*E^* \otimes x'^\alpha.
\]
Comparing determinants, we must have \(x'^{\alpha r} = 0 \implies x'^\alpha = 0\) since \((r, a) = 1\). Therefore, \(\alpha = 0\). Now, using the isomorphism
\[
\beta^*E^* \to E^*
\]
and letting
\[
q : A' \to A'/\langle \beta \rangle
\]
be the projection determined by \(\beta\), we obtain that \(E^*\) is a pullback from the quotient. Evaluating Euler characteristics, we obtain that \(\chi(E^*) = \frac{a}{b}\) is divisible by the order of \(\beta\). But \(\text{ord}(\beta)|b\) and \((b, \chi) = 1\), hence \(\text{ord}(\beta) = 1\) and \(\beta = 1\).

We are now in the position to calculate the normal bundles \(N_z\). Each isotypical component corresponds to a nontrivial pair \((\alpha, \beta)\). Each isotypical component has dimension \(2r\chi^a := \ell + 2\). We just argued the Chern roots are
\[
0, 0, x_1, \ldots, x_\ell,
\]
with \(x_i\) the roots of the tangent bundle of \(K_{w'}(A')\). The trivial weights come from the base of the Albanese map.

**Step 5.** With this understood, we calculate
\[
\left( \prod_{z \neq 1} \text{ch}_{-\langle \xi^{-1}, z \rangle} N^y_z \right)^{-1} = \prod_{\xi \neq 1} \left( 1 - \xi \right)^{-2} \prod_{i=1}^\ell \left( 1 - \xi e^{-x_i} \right)^{-1} = \frac{1}{\delta^2} \prod_{i=1}^\ell \frac{1 - e^{-x_i}}{1 - e^{-\delta x_i}},
\]
where \(\xi = \langle \xi^{-1}, z \rangle\) runs through the non-trivial \(\delta\)-roots of 1.

We now claim that
\[
i^*_Z \Theta_w \cong \Theta^\delta_{w'},
\]
where the last bundle carries a trivial $\mathbb{Z}$-linearization. Indeed, let $\Theta' \to A''$ be the polarization such that

$$p^*\Theta' = \Theta'' \implies \chi(\Theta'') = a.$$

Also, write $\Theta' = \pi^*\Theta''$, hence $\chi(\Theta') = ab$. We introduce the Mukai vectors

$$w'' = \left( r, \frac{k}{a}, \frac{\Theta''}{a} \right) \text{ over } A'', \text{ and}$$

$$w' = \left( r, \frac{k}{ab}, \frac{\Theta'}{a} \right) \text{ over } A'.$$

Observe that

$$p^*w'' = w, \pi^*w'' = bw'.$$

These equalities imply

$$i_1^*\Theta_w = \Theta_{w''}^a, i_2^*\Theta_{w''} = \Theta_{w'}^b,$$

giving the claim. Here, we factored $i = i_1 \circ i_2$ where

$$i_1 : \mathcal{M}_{w''}(A'') \to \mathcal{M}_{w}(A), \quad i_2 : \mathcal{M}_{w'}(A') \to \mathcal{M}_{w''}(A'')$$

are induced by pullback by $p$ and pushforward by $\pi$. Finally, the equivariant identification is implied by the discussion ending Subsection 5.2.

We calculate

$$\text{Trace}(\zeta, H^0(K_v, \Theta_w)) = \delta^3 \int_{K_v(A')} \prod_{i=1}^\ell \frac{x_i}{1 - e^{-x_i}} \cdot \prod_{i=1}^\ell \frac{1 - e^{-x_i}}{1 - e^{-\delta x_i}} \cdot \frac{1}{\delta^2} \cdot \text{ch}(\Theta_{w'}^\delta).$$

$$= \delta \int_{K_v(A')} \prod_{i=1}^\ell \frac{x_i}{1 - e^{-\delta x_i}} \cdot \text{ch}(\Theta_{w'}^\delta).$$

$$= \delta \chi(K_v', \Theta_{w'}) = \delta \frac{d_v^2}{d_v + d_{w'}} \left( \frac{d_{w'} + d_{w'}}{d_v} \right).$$

The last line follows from the backward application of Hirzebruch-Riemann-Roch. The proof is completed by observing that

$$d_{w'} = \frac{d_v}{\delta} \text{ and } d_{w''} = \frac{d_w}{\delta}.$$

5.4. The calculation of the Verlinde bundles. We are now in the position to confirm Conjecture 2 in degree 0, continuing to assume (A.1) and (A.2)'. We prove

**Theorem 3.** We have

$$\mathbf{E}(v, w) = \left( W_{-\frac{\chi}{\alpha}} \boxtimes W_{v}^{\dagger} \right) \otimes P^{-h} \otimes \bigoplus_{\zeta} \ell_{m^\zeta}.$$
The line bundles $\zeta \to \hat{A} \times A$ have order dividing $(a, b)$. An element $\zeta$ of order exactly $\omega$ comes with multiplicity

$$m_\zeta = \frac{1}{d_v + d_w} \sum_{\delta | ab} \delta^4 \left\{ \frac{ab/\omega}{\delta} \right\} \left( \frac{d_v}{\delta} + \frac{d_w}{\delta} \right).$$

Recall that in the above, for each line bundle $\zeta$ of order $(a, b)$ over $A \times \hat{A}$, we fix a root $\ell$ such that

$$\left( \chi, \frac{r}{a}, \frac{b}{b} \right) \ell = \zeta.$$

Each $\ell$ corresponds to a character of $A[-\chi] \times \hat{A}[r]$ which is uniquely defined only up to characters of $A[-\chi/a] \times \hat{A}[r/b]$.

**Proof.** The proof of the theorem follows the strategy laid out in [O2]. We give the relevant details here. First, we observed in (23) that

$$(-\chi, r)^* E(v, w) = H^0(K_v, \Theta_w) \otimes (\Theta^{-\chi} \boxtimes \hat{\Theta}^{-r})^k \otimes \left( (-\chi, r)^* P^{-h} \right).$$

This identifies $E(v, w)$ up to torsion $(-\chi, r)$-line bundles. In [O2] we observed the following equivariant identifications

$$(-\chi)^* W_{-\chi, \frac{k}{a}, \frac{k}{b}} = (\Theta^{-\chi})^k \boxtimes R_1$$

and

$$r^* W_{\frac{k}{a}, \frac{k}{b}} = \left( \hat{\Theta}^r \right)^{-k} \boxtimes R_2$$

for $R_1$ a representation of $H[-\chi]$ of dimension $(\chi/a)^2$ and central weight $-k$, while $R_2$ is a representation of $\hat{H}[r]$ of dimension $(r/b)^2$ and central weight $k$. Therefore,

$$(-\chi, r)^* \left( W_{-\chi, \frac{k}{a}, \frac{k}{b}} \boxtimes W_{\frac{k}{a}, \frac{k}{b}} \right) = (\Theta^{-\chi} \boxtimes \hat{\Theta}^{-r})^k \boxtimes R$$

where

$$R = R_1 \boxtimes R_2$$

is the product representation of $H[-\chi] \times \hat{H}[r]$. It suffices to explain that $H[-\chi] \times \hat{H}[r]$-equivariantly we have

$$H^0(K_v, \Theta_w) = R \otimes \bigoplus_\zeta \ell^{m_\zeta}$$

where $\ell$ runs over the characters of $A[-\chi] \times \hat{A}[r]$ modulo those which restrict trivially to the subgroup $A[-\chi/a] \times \hat{A}[r/b]$.

We make use of the morphism of Theta groups

$$H[a] \times \hat{H}[b] \to H[-\chi] \times \hat{H}[r]$$
which restricts to

\( (\alpha, \beta) \to (\alpha^{-\chi/a}, \beta r/b) \)

over the center

\( \mathbb{C}^* \times \mathbb{C}^* \hookrightarrow H[a] \times \hat{H}[b] \).

Furthermore, passing to the finite Heisenberg, two \( \tilde{H}[-\chi] \times \tilde{H}[r] \)-modules with the central weight \((-k, k)\) are isomorphic if and only if they are isomorphic as representations of the abelian group \( A[a] \times \hat{A}[b] \), see [O2]. Therefore, it suffices to establish the identification (24) equivariantly for the action of \( A[a] \times \hat{A}[b] \) on both sides.

Crucially, it was explained in Section 3.2 of [O1] that \( R_1 \) and \( R_2 \) are the trivial representations of \( A[a] \) and \( \hat{A}[b] \). For \( \zeta \) of order exactly \( \omega \) dividing \( (a, b) \), we compute

\[
m_\zeta = \frac{1}{\dim R} \cdot \frac{1}{a^n b^m} \sum_{\pi \in A[a] \times \hat{A}[b]} \langle \zeta, \pi^{-1} \rangle \text{Trace} \left( \pi, H^0(K_v, \Theta_w) \right)
\]

\[
= \frac{1}{a^2 b^2} \sum_{\delta(a, b)} \frac{1}{d_v + d_w} \left( \frac{d_v}{\delta} + \frac{d_w}{\delta} \right) \left( \sum_{\text{ord } (\pi) = \delta} \langle \zeta, \pi^{-1} \rangle \right)
\]

The proof is completed by Lemma 4 of [O1] which gives the sum

\[
\sum_{\text{ord } (\pi) = \delta} \langle \zeta, \pi^{-1} \rangle = \delta^4 \left\{ \frac{ab/\omega}{\delta} \right\}
\]

\[\Box\]

**Example 7.** Rank 1. Let

\[ v = (1, 0, -n), \ w = (1, k\Theta, n) \]

with \( n \) odd and \( k \geq 1 \). Then, by Example 1, we have

\[ M_v \cong A[n] \times \hat{A}, \ \alpha = (-a, 1), \]

and

\[ \Theta_w = (\Theta^k)_n \boxtimes \hat{\Theta}^{-k} \otimes (a, 1)^* \mathcal{P}. \]

Therefore,

\[ E(v, w) = a_* \left( (\Theta^k)_n \right) \boxtimes \hat{\Theta}^{-k} \otimes \mathcal{P}^{-1} \]

Theorem 1 is equivalent to

\[ a_* \left( (\Theta^k)_n \right) = \bigoplus_\zeta W_{\zeta, n, k} \otimes \mathcal{E}^\mathcal{m}_\zeta, \]
where $\zeta$ are line bundles over $A$ of order $\omega$ dividing $a = \gcd(n,k)$, $\ell$ is a root of $\zeta$ of order $\frac{n}{a}$, and

$$m_\omega = \frac{1}{k^2} \sum_{\delta | a} \frac{\delta^4}{a^2} \left( \frac{a/\omega}{\delta} \right) \left( \frac{k^2/\delta}{n/\delta} \right).$$

To apply the theorem, we invoke the result of Scala who proved the vanishing of higher cohomology of the tautological bundle, under the assumption that $k \geq 1$, cf. Theorem 5.2.1 [Sc].

5.5. Fourier-Mukai symmetries. We can now prove the Fourier-Mukai comparison, which may be seen as evidence for the strange duality conjecture for abelian surfaces. We establish:

**Theorem 4.** When $c_1(v)$ and $c_1(w)$ are divisible by the ranks $r$ and $r'$, we have

$$E(v,w)^\vee \cong \hat{E}(w,v).$$

**Proof.** The proof is immediate. We showed in Lemma 12 that the theorem is implied by Conjecture 2. However, we already proved the Conjecture under our assumptions. Indeed, Theorem 3 takes care of the degree 0 case. We reduce to degree 0 after tensoring by line bundles, which is possible since $c_1$ is divisible by the rank. The conjecture is still true by Lemma 13. □

**Appendix: Torsion points on abelian varieties**

In this appendix, we collect elementary arithmetic results used repeatedly throughout the paper. We begin by proving:

**Lemma 14.** Assume $(a,b,c,d)$ are coprime integers with $b^2 \equiv ac \mod d$. We can find $(m,n)$ coprime integers such that

$$cm \equiv bn \mod d, \ bm \equiv an \mod d.$$

**Proof.** We prove that $m = (a,b,d)m'$ and $n = (b,c,d)$ work for an appropriate choice of $m'$. We have

$$cm = bn \mod d \iff c(a,b,d)m' \equiv b(b,c,d) \mod d \iff \frac{c}{(b,c,d)^{m'}} \equiv \frac{b}{(a,b,d)} \mod \frac{d}{(b,ac,d)}.$$

Similarly, we have

$$bm \equiv an \mod d \iff b(a,b,d)m' \equiv a(b,c,d) \mod d.$$
We can find $B$, $C$ and $D$ such that
\[ bB + cC + dD = (b, c, d). \]
We let
\[ m' = \frac{bC + aB}{(a, b, d)} \mod \frac{d}{(b, ac, d)}. \]
The condition $b^2 \equiv ac \mod d$ is used to check that the pair $(m, n)$ thus obtained satisfies both congruences.

It suffices to show we can find $m'$ such that $(m', (b, c, d)) = 1$. This can be arranged provided
\[ \left( \frac{bC + aB}{(a, b, d)}, \frac{d}{(b, ac, d)}, (b, c, d) \right) = 1. \]
This is indeed the case, since if $p$ is a prime dividing all three numbers above, we can find $\beta, \gamma, \delta \geq 1$ such that
\[ p^\beta \| b, \ p^\gamma \| c, \ p^\delta \| d. \]
We must have $(p, a) = 1$ because $\gcd(a, b, c, d) = 1$. Since
\[ p|bC + aB \implies p|aB \implies p|B. \]
Moreover
\[ p|\frac{d}{(b, ac, d)} \implies \delta > \min(\beta, \gamma). \]
Now,
\[ b^2 \equiv ac \mod d \implies p^\delta|b^2 - ac \implies \beta < \gamma. \]
Indeed, if $\beta \geq \gamma$, we have $\delta \geq \gamma + 1$ and $2\beta \geq \gamma + 1$. But then $p^\delta$ does not divide $b^2 - ac$. Thus
\[ \delta > \beta, \gamma > \beta. \]
Finally,
\[ p^\beta \|(b, c, d) = bB + cC + dD \]
but the right hand side is divisible by $p^{\beta+1}$ since $c, d$ are, and $B$ is divisible by $p$. This contradiction completes the proof. \hfill \Box

**Lemma 15.** Assume $a, b, c, d$ are integers such that
\[ b^2 \equiv ac \mod d, \]
and
\[ \gcd\left( a, c, d, \frac{b^2 - ac}{d} \right) = 1. \]
Then, we can find exactly \( d^4 \) pairs \((x, y) \in A[d] \times A[d] \) such that
\[
ax = by, \quad cy = bx.
\]

**Proof.** We prove the Lemma in several steps. Fix \( a, b, c \). We let \( S_d \) be the set of solutions of the equations
\[
ax = by, \quad bx = cy, \quad dx = dy = 0,
\]
and let us write
\[
s_d = \#S_d.
\]
We show that \( s_d = d^4 \).

**Step 1.** We prove \( s_d \) is multiplicative. Assume \( d = d_1d_2 \) with \((d_1, d_2) = 1\). Note that
\[
\begin{pmatrix} a, c, d, \frac{ac - b^2}{d} \end{pmatrix} = 1 \implies \begin{pmatrix} a, c, d_i, \frac{ac - b^2}{d_i} \end{pmatrix} = 1, \quad 1 \leq i \leq 2.
\]
We claim that
\[
S_d \rightarrow S_{d_1} \times S_{d_2}, \quad (x, y) \rightarrow ((d_2x, d_2y), (d_1x, d_1y))
\]
is a bijection. Indeed, if we pick \( A \) and \( B \) such that
\[
Ad_1 + Bd_2 = 1,
\]
an inverse is given by
\[
((z_1, w_1), (z_2, w_2)) \rightarrow (Bz_1 + Az_2, Bw_1 + Aw_2).
\]
We conclude that
\[
s_d = s_{d_1}s_{d_2}.
\]

**Step 2:** We prove the lemma when \((a, b, c, d) = 1\). In fact, it suffices to assume \( d = p^\delta \) for some prime \( p \), by the step above.

We pick \( m \) and \( n \) as in Lemma 14:
\[
am \equiv bn \mod d, \quad cn \equiv bm \mod d.
\]
Since \((m, n) = 1\) then either
\[
(m, d) = 1 \text{ or } (n, d) = 1.
\]
Without loss of generality assume the former case holds, and write
\[
x = mz, \quad \text{for } z \in A[d].
\]
Set \( w = y - nz \in A[d] \). Clearly, we have
\[
bw = b(y - nz) = by - amz = by - ax = 0,
\]
\[
cw = c(y - nz) = cy - bmz = cy - bx = 0,
\]


\[ dw = 0. \]

We conclude \((b, c, d)w = 0\). Now, the mapping
\[ A[d] \times A[(b, c, d)] \to S_d, (z, w) \to (mz, nz + w) \]
is surjective and its kernel consists in pairs \((z, w)\) with
\[ mz = 0, nz + w = 0. \]

Since \((m, n) = 1\), writing \(1 = mA + nB\), we have
\[ z = -Bw. \]

Thus the kernel of the map is isomorphic to \(A[(b, c, d)]\), showing \(s_d = d^4\).

**Step 3:** Next we prove the statement when \(d = b^2 - ac\). Set \(e = (a, b, c)\) and write
\[ a = ea', b = eb', c = ec', (a', b', c') = 1. \]

We need to solve
\[ a'x - b'y = \alpha, b'x - c'y = \beta, \]
where \((\alpha, \beta) \in A[e]\) runs over all \(e^8 = e^4 \cdot e^4\) pairs of \(e\)-torsion points on \(A\).

We consider the morphism
\[ f : A \times A \to A \times A, \]
given by
\[ f(x, y) = (a'x - b'y, b'x - c'y). \]

By the previous step, we know the kernel of \(f\) has \((a'c' - b'^2)^4\) elements since \((a', b', c') = 1\).

Therefore, for each of the \(e^8\) choices of pairs \((\alpha, \beta)\), the equation
\[ f(x, y) = (\alpha, \beta) \]
has \((a'c' - b'^2)^4\) solutions. We obtain
\[ e^8(a'c' - b'^2)^4 = (ac - b^2)^4 = d^4 \]
possibilities for \((x, y)\).

**Step 4.** We claim that for each \(d\) as in the statement of the lemma, we have
\[ s_d \leq d^4. \]

By the first step, it suffices then to assume \(d = p^\delta\), for some prime \(p\). We will induct on \(\delta\), the base case \(\delta = 0\) being clear. Let \(\delta \geq 2\), the case \(\delta = 1\) being in fact simpler.

We may assume \(p\) divides \(a, b, c\) since otherwise \((a, b, c, d) = 1\), a case which has already been discussed. Write
\[ a = pa', b = pb', c = pc' \]
and note that
\[ d\mid ac - b^2 = p^2(a'c' - b'^2) \implies p^{\delta - 2}a'c' - b'^2. \]
Write \( d' = p^{\delta - 2} \). Since \((a, c, d, \frac{ac - b^2}{d}) = 1\) we know that \(a'c' - b'^2\) is not divisible by \(p^{\delta - 1}\).
In particular
\[ d'|a'c' - b'^2, \quad \left( a', c', d', \frac{a'c' - b'^2}{d'} \right) = 1. \]
Furthermore,
\[ (d, a'c') = p^{\delta - 2} = d', \]
so we can write
\[ d' = dU + (b'^2 - a'c')V. \]

For pairs \((x, y) \in S_d\), we must have
\[ a'x = b'y + \alpha, \quad b'x = c'y + \beta, \quad dx = dy = 0, \]
where \((\alpha, \beta) \in A[p] \times A[p]\). In fact, we obtain
\[ (b'^2 - a'c')x = b'\beta - c'\alpha, \quad dx = 0 \implies d'x = V(b'\beta - c'\alpha). \]
For each one of the \(p^4 \cdot p^4 = p^8\) pairs \((\alpha, \beta)\), we have at most \((d')^4\) choices for \((x, y)\) solving
\[ a'x = b'y + \alpha, \quad b'x = c'y + \beta, \quad d'x = V(b'\beta - c'\alpha), \quad d'y = U(a'\beta - b'\alpha). \]
Indeed, if a solution \((x_0, y_0)\) exists to begin with, then for all the other pairs we have
\[ (x - x_0, y - y_0) \in S_{d'}(a', b', c') \]
which has at most \((d')^4\) elements by induction. Therefore
\[ s_d \leq p^8 d'^4 = d^4, \]
as claimed.

Step 5. We can now prove that \(s_d = d^4\) for all \(d\) as in the lemma. Write
\[ ac - b^2 = p_1^{\alpha_1} \cdots p_r^{\alpha_r} q_1^{\beta_1} \cdots q_l^{\beta_l}, \]
where the primes \(p_1, \ldots, p_r\) also divide \((a, c)\). Now clearly \(d\) is product of \(p\)'s and \(q\)'s, but if \(p_i\) divides \(d\) then in fact
\[ p_i^{\alpha_i} | d, \]
by the gcd condition in the lemma. Thus we may write
\[ d = p_1^{\alpha_1} \cdots p_s^{\alpha_s} q_1^{\gamma_1} \cdots q_l^{\gamma_l}, \quad \text{for } s \leq r. \]
We have already observed that
\[ s_{p_i^{\alpha_i}} \leq p_i^{4\alpha_i} \]
in the previous step. Also,
\[ s_{\gamma_i} = q_i^4 \]
by the fact that \((a, c, q_i^{\gamma_i}) = 1\) and Step 2. Thus by the first step
\[ s_{ac-b^2} = s_{p_1^{\alpha_1}} \cdots s_{p_r^{\alpha_r}} \cdot s_{q_1^{\beta_1}} \cdots s_{q_l^{\beta_l}} \leq (p_1^{\alpha_1} \cdots p_r^{\alpha_r} q_1^{\beta_1} \cdots q_l^{\beta_l})^4 = (ac - b^2)^4. \]
Since we proved in the third step that equality occurs, we conclude that
\[ s_{p_i^{\alpha_i}} = p_i^{4\alpha_i} \]
and similarly for the powers \(q_i^{\gamma_i}\). By the first step again, we must have \(s_d = d^4\).

\[ \square \]

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