Math 20D - Fall 2008 - Final Exam

(A.) **Modelling** with differential equations:
- (i) savings account and mixture problems: \( \frac{dQ}{dt} = \text{rate in} - \text{rate out} \).

(B.) First order equations
- (i) Linear equations:
  \[ y' + p(t)y = q(t). \]
  * Solve by **integrating factors**. Bring the equation in standard linear form;
  * Integrating factor
  \[ u(t) = \exp \left( \int p(t) \, dt \right) \]
  * Multiply by \( u \), rewrite the equation as
  \[ (uy)' = uq. \]
- (ii) Nonlinear equations
  * **Separable**
  \[ \frac{dy}{dx} = f(x)g(y). \]
  Separate variables, then integrate.
  * **Autonomous equations**
  \[ \frac{dy}{dx} = f(y). \]
  Equilibrium solutions are the roots of \( f \).
  * **Exact**
  \[ M(x, y) + N(x, y)y' = 0. \]
  Check exactness:
  \[ M_y = N_x. \]
  Find a function \( f \) such that \( f_x = M, f_y = N \). Set \( f = \text{constant} \).

(C1.) Second order **homogeneous equations**
- (i) General facts:
  * **superposition**: if \( y_1, y_2 \) are solutions, \( c_1y_1 + c_2y_2 \) is also a solution.
  * **fundamental pair of solutions**: the Wronskian
  \[ W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \neq 0. \]
  For a fundamental pair, the general solution is
  \[ y = c_1y_1 + c_2y_2. \]
  * **Abel’s theorem**
  \[ W(y_1, y_2) = C \exp \left( - \int p(t) \, dt \right). \]
- (ii) **Constant coefficient equations**: \( p(t) = p, q(t) = q \).
  * Characteristic equation \( r^2 + pr + q = 0 \).
  * If the roots \( r_1, r_2 \) are real, then the fundamental solutions are
  \[ y_1 = e^{r_1t}, y_2 = e^{r_2t}. \]
* Complex roots: fundamental solutions are the real and imaginary part of $e^{rt}$. If $r_1 = \alpha + i\beta$, then
  \[
y_1 = e^{\alpha t} \cos \beta t, \quad y_2 = e^{\alpha t} \sin \beta t.
  \]
* Repeated roots $r_1 = r_2 = \alpha$, fundamental solutions
  \[
y_1 = e^{\alpha t}, \quad y_2 = te^{\alpha t}.
  \]

(C2) **Inhomogeneous second order** equations.
(i) General solution
  \[
y = y_p + y_h,
  \]
  where $y_p$ is the particular solution, $y_h$ is the homogeneous solution.
(ii) Find a particular solution by **undetermined coefficients**
  \[
y'' + py' + qy = g(t).
  \]
  Three cases: $g(t)$ can be exponential, trigonometric, polynomial.
  * For $g(t)$ polynomial, look for $y_p$ as a polynomial with undetermined coefficients. Try to
guess its degree first.
  * For trigonometric $g(t)$, look for $y_p = A \cos t + B \sin t$.
  * For exponential case $g(t) = e^{at}$, use
    \[
y_p = \frac{e^{at}}{f(a)}
    \]
    with $f(a) = a^2 + pa + q$. If $f(a) = 0$, look for $y_p = e^{at} (A + Bt)$ for undetermined $A, B$.
  * For a term $g(t) = e^{at} \times$ polynomial or trigonometric function, substitute $y = e^{at}u$, find
    the differential equation for $u$ using the $D$-notation, then solve for $u$ by undetermined
    coefficients.
(iii) Alternatively, you may use **variation of parameters**
  \[
y = u_1(t)y_1(t) + u_2(t)y_2(t)
  \]
  where
  \[
u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)} dt, \quad u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)} dt.
  \]

(D) **First order systems** of equations $x' = Ax$.
(i) Find **eigenvalues** $\lambda$ of $A$:
  \[
det(A - \lambda I) = 0
  \]
or alternatively for $2 \times 2$ matrices
  \[
\lambda^2 - \lambda \text{Tr}A + \det A = 0.
  \]
  **Eigenvectors** are found by solving the system
  \[
(A - \lambda I)v = 0.
  \]
(ii) Finding solutions: find eigenvalues $\lambda_1, \lambda_2$ with eigenvectors $v_1$ and $v_2$. General solution
  \[
x = c_1 e^{\lambda_1 t}v_1 + c_2 e^{\lambda_2}v_2.
  \]
(iii) Repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda$.
  * **Defective case**: one eigenvector $v$. Solutions
    \[
x_1 = e^{\lambda t}v, \quad x_2 = e^{\lambda t}(\alpha + tv)
    \]
    where $\alpha$ is a generalized eigenvector
    \[
(A - \lambda I)\alpha = v.
    \]
(iv) **Phase portraits**.
  * saddles: real eigenvalues of opposite sign,
  * nodes (sink or source): real distinct eigenvalues of same sign,
* spiral (sink or source) complex eigenvalues. To find the direction of spirals compute the velocity vector at a point on the trajectory.
* improper nodes: repeated eigenvalues.

(v) **Fundamental matrix**

\[ \Psi(t) = [x_1 \ x_2] . \]

The Wronskian is the determinant of the fundamental matrix

\[ W(x_1, x_2) = \det \Psi(t). \]

The normalized fundamental matrix / **matrix exponential**:

\[ \Phi(t) = \Psi(t)\Psi(0)^{-1} = e^{At}. \]

Solution to IVP \( x' = Ax \) and \( x(0) = x_0 \) is

\[ x = e^{At}x_0. \]

(vi) **Decoupling systems** and diagonalization. Let

\[ T = \Psi(0) = [v_1 \ v_2] . \]

Then

\[ T^{-1}AT = D \]

is diagonal. Applications:
* Solve systems

\[ x' = Ax + b. \]

Look for \( x = Ty \) where \( y \) satisfies the decoupled system

\[ y' = Dy + T^{-1}b. \]

(E) **Series solutions.**

(i) Radius of convergence.
(ii) Finding the recurrence relations between the coefficients.

(F) **Laplace transform.**

(i) When \( f \) has exponential growth,

\[ f \sim F(s) = \int_0^\infty e^{-st} f(t) \, dt. \]

(ii) Laplace transforms of the standard functions:

\[ 1 \sim \frac{1}{s}, \quad t^n \sim \frac{n!}{s^{n+1}}, \quad e^{at} \sim \frac{1}{s-a}, \quad e^{at}f(t) \sim F(s-a) \]

\[ \sin(at) \sim \frac{a}{s^2+a^2}, \quad \cos(at) \sim \frac{s}{s^2+a^2}. \]

(iii) Solving differential equations with Laplace transform:

\[ f' \sim sF(s) - f(0), \]

\[ f'' \sim s^2F(s) - sf(0) - f'(0). \]

(iv) Discontinuous functions:

\[ u_a(t) \sim \frac{e^{-sa}}{s}, \quad u_a(t)f(t-a) \sim e^{-sa}F(s). \]