Problem H-1. In this problem, we will analyze the depth of the decision tree in Binary Search (Algorithms 1.5 and 2.1):

Inputs: Positive integer $n$, a sorted array of keys $S[1..n]$, and a key $x$.
Output: The location of $x$ in the array, or 0 if it’s not in the array.

Here is the decision tree for the input array $S = \{10, 20, 30, \ldots, 150\}$ with $n = 15$:

Let $m$ be the maximum depth of the tree (in the example, $m = 4$). Answer all the questions on this page for the general case, not for this specific example.

The number of comparisons required to find a key $x$ is its depth $k$ if it’s in the tree, or $m$ or $m − 1$ if it’s not. The running time is roughly proportional to this number.

We will only consider searches for keys that are in $S$, and in (a)–(e) we will assume the bottom level is “full” (all leaves occur on level $m$; if it’s not full, some are on level $m$ and some on level $m − 1$).

(a) What is the number of nodes at depth $k$, where $1 \leq k \leq m$ are as indicated in the diagram?

(b) Express the total number of nodes $n$ as a function of $m$ by summing up the answers for (a). Convert it to closed form by plugging suitable values into the geometric series

$$1 + r + r^2 + \cdots + r^j = \frac{1 - r^{j+1}}{1 - r}.$$

(c) Assume you are searching for a key that is present in the list $S$, and that all $n$ keys are equally likely. What is the expected depth of the node containing $x$? You may use the sum

$$1 + 2r + 3r^2 + \cdots + jr^{j-1} = d \frac{d}{dr} (1 + r + r^2 + \cdots + r^j) = d \frac{d}{dr} \frac{1 - r^{j+1}}{1 - r} = \frac{r^j(j(r-1)-1)+1}{(1-r)^2}.$$

(The values plugged into it may differ from part (b).) Your final answer should be an exact answer.

(d) Reexpress your answer to (c) solely in terms of $n$ (use the answer to (b) to determine how to substitute for $m$ in terms of $n$).

(e) Give a simplified asymptotic form of the final answer to (d).

(f) Next we compute the expected depth recursively, for all $n$, without restricting to when the bottom level is full. Prove that the expected number of comparisons to locate a key is given by the recursion

$$A(0) = 0 \\
A(1) = 1 \\
A(n) = \frac{1}{n} \left( \left\lfloor \frac{n-1}{2} \right\rfloor \left(1 + A\left( \left\lfloor \frac{n-1}{2} \right\rfloor \right)\right) + \left\lfloor \frac{n}{2} \right\rfloor \left(1 + A\left( \left\lfloor \frac{n}{2} \right\rfloor \right)\right) + 1 \right) \quad \text{for } n > 1.$$