1. Koepf 2.11(b) The sum starts at $k = 1$ instead of $k = 0$. Shift it so the $k = 0$ term is nonzero:

$$\ln(1 + x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}$$

Now the initial term is $t_0 = x$, and the term ratio is

$$r(k) = \frac{(-1)^{k+1} x^{k+2}}{k+2} / \frac{(-1)^{k} x^{k+1}}{k+1} = \frac{k+1}{k+2} \cdot \frac{x}{(-x)} = \frac{(k+1)(k+1)}{(k+2)(k+1)}$$

so

$$\ln(1 + x) = x \cdot {}_2F_1 \left[ \frac{1, 1}{2} \mid -x \right]$$

Koepf 2.11(f) The initial term is $t_0 = x$, and the term ratio is

$$r(k) = \frac{(-1)^{k+1} x^{2k+3}}{2k+3} / \frac{(-1)^{k} x^{2k+1}}{2k+1} = \frac{2k+1}{2k+3} \cdot \frac{x^2}{(-x^2)} = \frac{(k+1/2)(k+1)}{(k+3/2)(k+1)}$$

so

$$\arctan(x) = x \cdot {}_2F_1 \left[ \frac{1/2, 1}{3/2} \mid -x^2 \right]$$

Note that $t_k \neq 0$ for integer $k < 0$. The bound $k = 0$ is unnatural; this manifests itself in our having to multiply by $\frac{x+1}{k+1}$ and then include a 1 as an upper parameter.

Koepf 2.9(d) The first binomial coefficient is 0 unless $0 \leq k \leq n$, and the second is 0 unless $n \leq 2k$, so the summation range is actually $[n/2] \leq k \leq n$.

The general term is $t_k = \frac{n! (2k)!}{k! (n-k)! (2k-n)!} = \frac{(2k)!}{k! (n-k)! (2k-n)!}$ so the term ratio is

$$r(k) = \frac{(2k+2)!}{(2k)!} \cdot \frac{k! (n-k)! (2k-n)!}{(k+1)! (n-k-1)! (2k+2-n)!} = \frac{(2k+2)(2k+1)(n-k)}{(2k+2-n)(2k+1-n)(k+1)} = \frac{(k+1/2)(k-n)}{(k+1-n/2)(k+(1-n)/2)}$$

Answer for non-integer $n$. This is what is produced by the software on the worksheet. The 0th term is $t_0 = \binom{n}{0} \binom{0}{0} = \binom{n}{0} = \frac{n!}{n!(-n)!} = \frac{1}{\Gamma(n+1)\Gamma(1-n)} = \frac{1}{n \Gamma(n) \Gamma(1-n)} = \frac{\sin(\pi n)}{\pi n}$ using the reflection formula for $\Gamma$ (Koepf, p. 6, (1.9)). Then the sum is

$$\binom{0}{n} {}_3F_2 \left[ \frac{-n, 1/2, 1}{(1-n)/2, 1-n/2} \mid -1 \right]$$

for $n \in \mathbb{C} \setminus \mathbb{N}$.

Of course, we’re mainly interested in positive integer $n$, so this won’t do. It’s wrong for positive integers $n$ because we multiply by $\binom{0}{n} = 0$, and then one of the denominator factors in the hypergeometric series will divide by 0 to compensate.
Non-negative integer \( n \), Method I. The initial term evaluates to \( t_0 = 0 \), so we must shift the sum. (We could have done this in advance, but this will show how to do it algorithmically.) There are two cases: \( n \) even and \( n \) odd. When \( n \) is even, we should shift the sum down by \( n/2 \), and when it’s odd, by \((n + 1)/2\). Algorithmically, step 4 of Koepf page 21 says to let \( m \) be the smallest integer \( \beta \) and then shift by \( 1 - m \). When \( n \) is even, \( m = 1 - n/2 \) and we shift by \( n/2 \) (i.e., the new sum index \( k' \) satisfies \( k = k' + n/2 \)), and when \( n \) is odd, \( m = (1 - n)/2 \) and we shift by \((n + 1)/2\). (In both cases, we have shifted by \([n/2]\) as we expect).

**even** \( n \): Set \( k = k' + n/2 \). The sum is
\[
\sum_{k'=0}^{n/2} \binom{n}{k'+(n/2)} \binom{2k'+n}{n}
\]
and the upper bound is natural, so we can replace it by \( \sum_{k'=0}^{\infty} \). The term ratio \( r_e \) in terms of the original ratio is
\[
r_e(k') = r(k' + n/2) = \frac{(k' + (n + 1)/2)(k' - n/2)}{(k' + 1)(k' + 1/2)}
\]
(notice the shift caused a \( k' + 1 \) denominator factor). The initial term is \( \binom{n}{n/2} \binom{n}{n} = \binom{n}{n/2} \), so the sum is
\[
\left[ \begin{array}{c} n \\ n/2 \end{array} \right] F_1 \left[ \frac{n+1}{2}, \frac{-n}{2} \bigg| \frac{1}{2} \right] - 1 \quad \text{for even } n \in \mathbb{N}.
\]

**odd** \( n \): Set \( k = k' + (n + 1)/2 \). The sum is
\[
\sum_{k'=0}^{(n+1)/2} \binom{n}{k'+(n+1)/2} \binom{2k'+n+1}{n}
\]
and the upper bound is natural, so we can replace it by \( \sum_{k'=0}^{\infty} \). The term ratio \( r_o \) is
\[
r_o(k') = r(k' + (n + 1)/2) = \frac{(k' + n/2 + 1)(k' + (1 - n)/2)}{(k' + 3/2)(k' + 1)}
\]
and the initial term is \( \binom{n}{(n+1)/2} \binom{n+1}{n} = \binom{n}{(n+1)/2} \), so the sum is
\[
n \left[ \begin{array}{c} n \\ (n+1)/2 \end{array} \right] F_1 \left[ \frac{1+n/2}{3/2}, \frac{-n}{2} \bigg| \frac{1}{2} \right] - 1 \quad \text{for odd } n \in \mathbb{N}.
\]

Method II. There were two cases for the lower bound of the sum, but the upper bound was always \( n \); so make a new sum \( \sum u_k' \) where \( u_k' = t_{n-k} \):
\[
\sum_{k'=0}^{[n/2]} \binom{n}{n-k'} \binom{2(n-k')}{n}
\]
instead. This sum can be taken as \( \sum_{k'=0}^{\infty} \). The new term ratio \( \tilde{r} \) in terms of the original is
\[
\tilde{r}(k') = u_{k'+1}/u_k' = t_{n-k'-1}/t_{n-k'} = 1/r(n-k'-1), \text{ so}
\]
\[
\tilde{r}(k') = -\frac{(n-k'-1+(1-n)/2)(n-k'-1+1-n/2)}{(n-k'-1-n)(n-k'-1+1/2)} = \frac{(k' + (1-n)/2)(k' - n/2)}{(k' + 1/2 - n)(k' + 1)}
\]
The new initial term is \( u_0 = \binom{2n}{n} \). So the whole sum is
\[
\left[ \begin{array}{c} 2n \\ n \end{array} \right] F_1 \left[ \frac{1-n}{2}, \frac{-n}{2} \bigg| \frac{1}{2} \right] - 1 \quad \text{for } n \in \mathbb{N}.
\]
2. Koepf 2.3

\[ F(x) = p F_q \left[ \begin{array}{c} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{array} \right] x = \sum_{k=0}^{\infty} \frac{(\alpha_1) \cdots (\alpha_p) k! x^k}{(\beta_1) \cdots (\beta_q) k!} \]

Now \( \theta x_k = x \cdot kx^{k-1} = kx^k \), so \( (\theta + c)x^k = (k + c)x^k \). Thus every term of the sum is an eigenvector of \( \theta + c \) for any constant \( c \). The left side of (2.23) is

\[ \theta(\theta + \beta_1 - 1) \cdots (\theta + \beta_q - 1) F(x) = \sum_{k=0}^{\infty} k \cdot (\beta_1 + k - 1) \cdots (\beta_q + k - 1) \frac{(\alpha_1) \cdots (\alpha_p) k!}{(\beta_1) \cdots (\beta_q) k!} x^k \]

and the right side of (2.23) is

\[ x(\theta + \beta_1 - 1) \cdots (\theta + \beta_q - 1) F(x) = \sum_{k=0}^{\infty} (\alpha_1 + k) \cdots (\alpha_p + k) \frac{(\alpha_1) \cdots (\alpha_p) k!}{(\beta_1) \cdots (\beta_q) k!} x^{k+1} \]

Just shift \( k \) by 1 to make them agree.

Koepf 2.4 Let \( n \) denote \( \alpha_1 \), and let \( \alpha_2, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \) be given. We have

\[ F_n(x) = p F_q \left[ \begin{array}{c} n, \alpha_2, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{array} \right] x \] and \( \theta F_n(x) = \sum_{k=0}^{\infty} \frac{(n) (\alpha_2) \cdots (\alpha_p) k!}{(\beta_1) \cdots (\beta_q) k!} x^k \)

while

\[ n(F_{n+1}(x) - F_n(x)) = n \sum_{k=0}^{\infty} ((n+1)k - (n)k) \frac{(\alpha_2) \cdots (\alpha_p) k!}{(\beta_1) \cdots (\beta_q) k!} x^k \]

Now \( (n+1)k = (n+1)(n+2) \cdots (n+k) = (n+k)(n+1)k_{k-1} \) and \( (n)k = n \cdot (n+1)k_{k-1} \), so \( (n+1)k - (n)k = k \cdot (n+1)k_{k-1} \). Combine this with the additional factor of \( n \) to get the result.

The proof for other \( \alpha \)’s is the same, and the proof for \( n = \beta_i \) is similar.

Koepf 2.5 Let’s get a recursion for \( \alpha_1 \). We may expand the left side of (2.23) as

\[ \sum_{r=1}^{q+1} e_{r-1}(\beta_1 - 1, \ldots, \beta_q - 1) \theta^r F_n(x) \] (*)

where \( e_k \) is the elementary symmetric function. The right side may be expanded similarly.

Let \( E \) be the shift operator \( E_n \). Then we make replacements

\[ \theta F_n = n(E - 1) F_n \]

\[ \theta^2 F_n = n(\theta F_{n+1} - \theta F_n) = n \left( (n+1)(F_{n+2} - F_{n+1}) - n(F_{n+1} - F_n) \right) \]

\[ = n \left( (n+1)F_{n+2} - (2n+1)F_{n+1} + n F_n \right) = \left( n(n+1)E^2 - (2n+1)E + n \right) F_n \]

and so forth into (*). The resulting equation will have order \( q + 1 \) on the left side and order \( p \) on the right side; combining all the terms onto one side will give an equation of order \( \max(p, q + 1) \). If \( p = q + 1 \) it might seem that the highest order terms could cancel, but on the left we have \( \theta^{q+1} \) and on the right, \( x \theta^p \), so the leading coefficients don’t cancel.
The denominator formulas are obtained similarly. We use $E^{-1}$ instead of $E$, but the same orders will be obtained.

3. $e^x$ Let $D = \frac{d}{dx}$, so $\theta = xD$. 2.3 gives $\theta F(x) = x F(x)$, so $xD F(x) = x F(x)$, and $DF(x) = F(x)$ unless $x = 0$. Thus, $(D - 1)F(x) = 0$; the $x = 0$ case is included by analyticity.

\sin x Let $G(x) = \sin x = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}$. Now, consider $z = Ax^r$ and $\theta = z \frac{d}{dz}$ (here $z = (-\frac{1}{2})x^2$). By the chain rule,

$$\theta = z \frac{d}{dz} = \frac{d}{dz} \frac{d}{dx} = \frac{Ax^r}{r A x^{r-1}} \frac{d}{dx} = \frac{x}{r \cdot dx}$$

which here is $\theta = \frac{x}{2} \frac{d}{dx}$. Then by Koepf #2.3, $F(x) = G(x)/x$ satisfies

$$\theta(\theta + 3/2 - 1) F = z F(x).$$

Putting it all in terms of $x$.

$$(x/2) D(xD/2 + 1/2) F = -\frac{x^2}{4} F \quad \text{so} \quad xD(xD + 1) F = -x^2 F$$

The left side is tedious to simplify, but straightforward. Move all $x$’s left and $D$’s right, using the rules $Dx = xD + 1$ and more generally, $Df = fD + f'$, where $x$ and $f$ are multiplication operators. Later we may use software for dealing with noncommutative operators, but this isn’t described in the textbooks we’re now using. Here’s the brute force way.

We have the operator

$$xD xD + xD = x(xD + 1)D + xD = x^2 D^2 + xD + xD = x^2 D^2 + 2xD$$

giving $(x^2 D^2 + 2xD + x^2)F = 0$. We actually want an equation for $G$ where $F = x^{-1}G$, so $(x^2 D^2 + 2xD + x^2)x^{-1}G = 0$. As operators,

$$Dx^{-1} = x^{-1} D - \frac{1}{x^2}$$

$$D^2 x^{-1} = Dx^{-1} D - D \frac{1}{x^2} = (x^{-1} D - \frac{1}{x^2} D - \frac{2}{x^3}) D = \frac{1}{x} D^2 - \frac{2}{x^2} D + \frac{2}{x^3}$$

Then as operators,

$$([x^2 D^2] + [2xD] + [x^2]) x^{-1} = \left[ xD^2 - 2D + \frac{2}{x} \right] + \left[ 2D - \frac{2}{x} \right] + [x] = xD^2 + x$$

so $(xD^2 + x) G(x) = (xD^2 + 1)G(x) = 0$. So when $x \neq 0$ we have $(D^2 + 1) G(x) = 0$, and by analyticity this holds at $x = 0$ too.

4. Koepf 2.21

(a) $(aq^n; q)_\infty = \prod_{j=0}^{\infty} (1-aq^j)$ and $(a; q)_\infty = \prod_{j=0}^{\infty} (1-aq^j)$, so $(a; q)_\infty = (aq^n; q)_\infty = \prod_{j=0}^{n-1} (1-aq^j) = (a; q)_n$.

(b) $\frac{(q\sqrt{a}; q)_n}{(\sqrt{a}; q)_n} = \frac{1 - \sqrt{a} \cdot q^n}{1 - \sqrt{a}}$. The same holds with $\sqrt{a} \to -\sqrt{a}$. Multiplying gives

$$\frac{(q\sqrt{a}; q)_n (-q\sqrt{a}; q)_n}{(\sqrt{a}; q)_n (-\sqrt{a}; q)_n} = \frac{(1 - \sqrt{a} \cdot q^n)(1 + \sqrt{a} \cdot q^n)}{(1 - \sqrt{a})(1 + \sqrt{a})} = \frac{1 - a q^{2n}}{1 - a}.$$
(c)\[ (a; q)_n (-a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j)(1 + aq^j) = \prod_{j=0}^{n-1} (1 - a^2(q^2)^j) = (a^2; q^2)_n \]

(d)\[ (a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j) = \prod_{j=0}^{n-1} (-aq^j)(1 - \frac{1}{aq^j}) \]

\[ = \left( \prod_{j=0}^{n-1} (-aq^j) \right) \left( \prod_{j=0}^{n-1} \frac{1 - q^j}{aq^{n-j}} \right) = (-a)^n q^{\sum_{j=0}^{n-1} j} (q^{1-n}/a; q)_n \]

\[ = (q^{1-n}/a; q)_n (-a)^n q^{(n)} \]

**Koepf 3.15(b)** The initial term is $t_0 = 1$. The term ratio is

\[ R(k) = \frac{\left[ \frac{n+1}{k+1} \right]_q x^{k+1}}{\left[ \frac{n}{k} \right]_q x^k} = \frac{\left( \frac{q; q)_n}{(q; q)_n (q; q)_{n-k}} \right)}{\left( \frac{q; q)_n (q; q)_{k+1}}{(q; q)_n (q; q)_{n-k-1}} \right)} x^{k+1} \]

\[ \frac{x^k}{1 - q^{k+1}} = \left( 1 - q^{-n}Q \right)^2 x = \left( \frac{1 - q^{-n}Q}{1 - q^{-n}Q} \right)^2 x = \left( \frac{1 - aQ}{1 - q^{-n}Q} \right) \left( \frac{1 - aQ}{1 - q^{-n}Q} \right) \cdot (q^{2n}x)^2 \cdot (-Q)^{-2} \]

where $a = q^{-n}$. It looks like it may be a $2\phi_1$, but we have the power $(-Q)^{-2}$. It is an $r\phi_s$ with $1 + s - r = -2$, so $s = 1$ is O.K., but we should increase $r$ to 4 by adding more 0 parameters in the numerator:

\[ R(k) = \frac{(1 - aQ)(1 - aQ)(1 - 0Q)(1 - 0Q)}{(1 - qQ)(1 - qQ)} (q^{2n}x)^2 (-Q)^{-2} \]

So the sum is

\[ 1 \cdot 4\phi_1 \left[ \begin{array}{cc} q^{-n}, q^{-n}, 0, 0 \\ 1 \end{array} \right] q, q^{2n}x \]