1. (a) We have
\[
\phi_s \left[ \frac{x}{\alpha_r} \mid \begin{matrix} \alpha_1, \ldots, \alpha_r \\ \beta_1, \ldots, \beta_s \end{matrix} \right] = \sum_{k=0}^{\infty} \frac{(\alpha_1, \ldots, \alpha_{r-1}; q)_k x^k \alpha_r^k}{(\beta_1, \ldots, \beta_s; q)_k} \left( \frac{(-1)^k q^{(k)}}{(q; q)_k} \right)^{1+s-r} \tag{1}
\]

The second fraction on the right is
\[
\frac{(\alpha_r; q)_k}{\alpha_r^k} = \prod_{j=0}^{k-1} \frac{1 - \alpha_r q^{j-1}}{\alpha_r} = \prod_{j=0}^{k-1} \left( \frac{1}{\alpha_r} - q^{j-1} \right)
\]
and as \( \alpha_r \to \infty \), this tends to
\[
\prod_{j=0}^{k-1} (-q^{j-1}) = (-1)^k q^{(k)}.
\]

The limit as \( \alpha_r \to \infty \) of (1) is then
\[
\sum_{k=0}^{\infty} \frac{(\alpha_1, \ldots, \alpha_{r-1}; q)_k x^k}{(\beta_1, \ldots, \beta_s; q)_k} \left( \frac{(-1)^k q^{(k)}}{(q; q)_k} \right)^{1+s-r} = \phi_s \left[ \alpha_1, \ldots, \alpha_{r-1} \mid \begin{matrix} \beta_1, \ldots, \beta_s \end{matrix} q, x \right].
\]

(b) The series form of the \( q \)-binomial theorem is
\[
\sum_{k=0}^{\infty} \frac{(\alpha; q)_k}{(q; q)_k} x^k = \frac{(\alpha x; q)_{\infty}}{(x; q)_{\infty}} \tag{2}
\]
Setting \( \alpha = 0 \) gives the equality for \( e_q(x) \).

For \( E_q(x) \), replace \( \alpha \) by \( 1/\alpha \) and \( x \) with \( -\alpha x \), and then set \( \alpha = 0 \).

(c) For the limits, we have that
\[
e_q(x(1-q)) = \sum_{k=0}^{\infty} \frac{(x(1-q))^k}{(q; q)_k} = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q}.
\]

and as \( q \to 1 \) the denominator turns into the ordinary \( k! \). The proof for \( E_q(x) \) is similar.

The two formulas \( e_q(x) = 1/(x; q)_\infty \) and \( E_q(x) = (x; q)_\infty \) imply \( e_q(x) E_q(-x) = 1 \).

For the trig function identities, we have
\[
\sin_q(x) \sin_q(x) + \cos_q(x) \cos_q(x) = \frac{1}{4} \left( -e_q(ix)E_q(ix) + e_q(ix)E_q(-ix) + e_q(-ix)E_q(ix) - e_q(-ix)E_q(-ix) \right)
+ e_q(ix)E_q(ix) + e_q(ix)E_q(-ix) + e_q(-ix)E_q(ix) + e_q(-ix)E_q(-ix)
= \frac{1}{2} \left( e_q(ix)E_q(-ix) + e_q(-ix)E_q(ix) \right) = \frac{1}{2} (1 + 1) = 1
\]
and the other is proved similarly.

2. (a) To find the complete solution to the recurrence equation
\[
f(n+3) - 8f(n+2) + 21f(n+1) - 18f(n) = 3^n \quad (n \in \mathbb{N}) \tag{3}
\]
we first find the homogeneous solution. The left side may be rewritten \((E-3)^2(E-2)f(n)\), so the homogeneous solution is \( f_h(n) = (an+b)3^n + c \cdot 2^n \) for some constants \( a, b, c \).
A particular solution of the form \( d \cdot 3^n \) won’t work because \( 3^n \) is included in the homogeneous solution; instead we must try \( d \cdot n^2 3^n \). (This is guaranteed to work, it’s just a matter of finding \( d \) now.) Plugging this into the equation gives

\[
(3^n \cdot ((n + 3)^2 \cdot 3^3 - 8(n + 2)^2 \cdot 3^2 + 21(n + 1)^2 \cdot 3 - 18n^2)d = 18d \cdot 3^n = 3^n
\]

so \( d = 1/18 \) and a particular solution is \( f_p(n) = n^2 3^n/18 \). The complete solution is

\[
f(n) = \left( \frac{n^2}{18} + an + b \right) 3^n + c \cdot 2^n \quad \text{for some constants } a, b, c.
\]

(b) Plug in the initial conditions:

\[
0 = f(0) = b + c
\]
\[
\frac{1}{6} = f(1) = \left( \frac{1}{18} + a + b \right) \cdot 3c
\]
\[
2 = f(2) = \left( \frac{4}{18} + 2a + b \right) \cdot 9 + 4c
\]

Solve the equations to get \( a = b = c = 0 \). The solution is then \( f(n) = \frac{n^2 3^n}{18} \).

(c) If \( n \in \mathbb{R} \), then every “residue class modulo 1” is independent of the others, so the “constants” \( a, b, c \) are replaced by any functions \( a(n), b(n), c(n) \) that have period 1. They don’t even have to be continuous functions.

(d) If both sides of the recurrence are multiplied by \( n - 100 \), then the original equation needn’t hold at \( n = 100 \). Thus, \( f(103) \) may be chosen arbitrarily. We will have the solution as given in (a) for \( n = 0, 1, \ldots, 102 \), and a solution of the same form for \( n \geq 103 \) with new constants \( a', b', c' \) that will depend upon \( a, b, c, f(103) \).

3. Sister Celine’s method. These problems are on the maple printout \texttt{hw2.mws}. 