5.1

\[ \Delta k^m = (k + 1)(k) \cdots (k - m + 2) - k(k - 1) \cdots (k - m + 2)(k - m + 1) \]

\[ = [(k + 1) - (k - m + 1)] \cdot k(k - 1) \cdots (k - m + 2) \]

\[ = m \cdot k(k - 1) \cdots (k - m + 2) = m \cdot k^{m+1} \]

so \( \Delta k^{m+1} = (m + 1)k^m \), and provided \( m \neq -1 \), dividing by \( m + 1 \) gives

\[ k^m = \frac{\Delta k^{m+1}}{m + 1} = \frac{(k + 1)^{m+1} - k^m}{m + 1} \]

Sum this for \( k = a, a + 1, \ldots, b - 1 \):

\[ \sum_{k=a}^{b-1} k^m = \sum_{k=a}^{b-1} \frac{(k + 1)^{m+1} - k^m}{m + 1} \]

\[ = \frac{1}{m + 1} \left( -a^m + (a + 1)^m - (a + 1)^m + (a + 2)^m - \cdots - (b - 1)^m + b^m \right) \]

\[ = \frac{b^m - a^m}{m + 1} \]

5.2 We have

\[ u_k \Delta v_k + v_{k+1} \Delta u_k = u_kv_{k+1} - u_kv_k + v_{k+1}u_k - v_{k+1}u_k = -u_kv_k + v_{k+1}u_k \]

and summing for \( k = a, \ldots, b - 1 \) [Note: the book has a typo] gives

\[ \sum_{k=a}^{b-1} (u_k \Delta v_k + v_{k+1} \Delta u_k) = u bv_b - u av_a = u kv_k \bigg|_{k=a}^b \]

by telescoping. This rearranges into

\[ \sum_{k=a}^{b-1} u_k \Delta v_k = u_k v_k \bigg|_{k=a}^b - \sum_{k=a}^{b-1} v_{k+1} \Delta u_k \]

All antidifferences of \( H_k = \sum_{j=1}^{k} \frac{1}{j} \) are the same up to additive constant, so we take

\[ s_n = \sum_{k=1}^{n-1} H_k = \sum_{k=1}^{n-1} \sum_{j=1}^{k} \frac{1}{j} = \sum_{j=1}^{n} \sum_{k=j}^{n-1} \frac{1}{j} \]

\[ = \sum_{j=1}^{n} \frac{n - j}{j} = n \sum_{j=1}^{n-1} \frac{1}{j} - \sum_{j=1}^{n-1} 1 = \left[ 1 - n + nH_{n-1} \right] \]

Actually, this didn’t make use of summation by parts at all . . .

5.10 The desired antidifference has the form

\[ s_k = C + \sum_{n=0}^{k-1} a_n = C + \sum_{n=0}^{k-1} (t_{n+m} - t_n) = C + (t_{k-1} + t_k + \cdots + t_{k+m-1}) - (t_0 + t_1 + \cdots + t_{m-1}) \]

for some constant \( C \). We are given that \( t_k \) is hypergeometric in \( k \), and we want \( s_k \) to be hypergeometric in \( k \) as well. The terms \( t_{k-1}, \ldots, t_{k+m-1} \) are in the same “rational similarity class” (their quotients are rational functions of \( k \)). The terms \( C + t_0 + \cdots + t_{m-1} \) are constant w.r.t. \( k \), and their ratio with \( t_k \) etc. will in general be hypergeometric but not rational, so the
In class we showed \( \frac{a(k+1)}{a(k)} \) which in Koepf’s notation is \( \frac{a_{k+1}}{a_k} = \frac{1+R_k}{R_{k+1}} \) and therefore in each problem,

\[
a_k = a_j \prod_{n=j}^{k-1} \frac{1+R_n}{R_{n+1}}
\]

where \( a_j \) is a suitable initial value. Then

<table>
<thead>
<tr>
<th>( R_k )</th>
<th>( \frac{a_{k+1}}{a_k} )</th>
<th>( a_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>( \frac{2\alpha-1}{\alpha} )</td>
<td>( a_0 \cdot \left( \frac{2\alpha-1}{\alpha} \right)^k )</td>
</tr>
<tr>
<td>( \alpha - 1 )</td>
<td>( \frac{1+k}{k+1} )</td>
<td>( a_0 )</td>
</tr>
<tr>
<td>( k )</td>
<td>( \frac{1+k^2}{(k+1)^2} )</td>
<td>( a_1 \prod_{n=0}^{k-1} \frac{1+n^2}{(n+1)^2} )</td>
</tr>
<tr>
<td>( k^2 )</td>
<td>( \frac{(k+1)!}{k!} )</td>
<td>( a_1 \prod_{n=1}^{k-1} \frac{(n+1)^2}{n} = a_1 \cdot \frac{k^2}{(k-1)!} = a_1 \cdot k \cdot k! )</td>
</tr>
</tbody>
</table>
| \( 1/k \) | \( \frac{(k+1)/k}{1/(k+1)} = \frac{(k+1)^2}{k} \) | Note: The denominator at \( n = 0 \) is 0, so we start with \( a_1 \).
| \( (k-1)/k \) | \( \frac{(2k-1)(k+1)}{k^2} \) | \( a_1 \prod_{n=1}^{k-1} \frac{(2n-1)(n+1)}{n^2} = a_1 \cdot \frac{(2k-3)!}{(k-1)!2^k} = a_1 \cdot \frac{(2k-3)!}{(k-1)!2^k} \) |
| \( (k+1)/k \) | \( \frac{(2k+1)(k+1)}{k(k+2)} \) | \( a_1 \cdot \frac{(2k-1)!/3 \cdot k!}{(k-1)! (k+1)/3} = a_1 \cdot \frac{(2k-1)!}{(k-1)! (k+1)!} = a_1 \cdot \frac{(2k)!/2^k}{(k-1)! (k+1)!} = a_1 \cdot \frac{2k}{(k-1)! (k+1)!} \) |

where we used the notation \( (2n)!! = 2 \cdot 4 \cdot \cdots \cdot (n-2)n \) and \( (2n+1)!! = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)(2n+1) \) for integer \( n \).

In Example 5.3, page 71 of Koepf, the term ratio for \( a(k) = \binom{n}{k} \) is computed:

\[
\frac{a(k+1)}{a(k)} = \frac{n-k}{k+1}.
\]

Since \( n \) is a symbol rather than a specific number, the numerator and denominator do not have roots differing by an integer, so Gosper’s algorithm finds “\( p, q, r \)” quickly and terminates. If \( n \) represents a specific integer, however, the algorithm proceeds differently.

Initially we choose \( p(k) = 1, q(k) = n-k+1 = -(k-(n+1)), r(k) = k \). Consider \( g_j(k) := \gcd(q(k), r(k+j)) = \gcd(k-(n+1), k+j) \). This is \( k+j \neq 1 \) when \( j = -(n+1) \); for \( j \) to be a nonnegative integer requires \( n = -1, -2, -3, \ldots \). Write this as \( n = -N, j = N-1, g_j(k) = k+N-1 \) with \( N \) positive. The algorithm computes a different \( p, q, r \), as follows:
We are given
\[ f(k) = q(k)/g(j) = -1 \]
and these new \( q, r \) are relatively prime at all shifts so this is final.

The next step is to find a polynomial \( f(k) \) satisfying
\[ q(k + 1)f(k) - r(k)f(k - 1) = p(k), \]
which becomes
\[ -f(k) - f(k - 1) = (k + N - 1)(k + N - 2)\ldots(k + 1) \tag{1} \]
This is a constant coefficient recursion with a polynomial inhomogeneity, so we expect that \( f(k) \) should be a polynomial of degree \( N - 1 \). However, we will do this as in Gosper’s algorithm. Since \( q \) and \( r \) have different leading terms, we are in “Case 1.” The degree bound for \( f(k) \) is then
\[ \deg(p) = \max\{\deg(q), \deg(r)\} = (N - 1) - 0 = N - 1 = -n - 1. \]
So we set \( f(k) \) to a generic polynomial of this degree:
\[ f(k) = \sum_{i=0}^{N-1} c_i k^i \]
and plug that into (1), collect in powers of \( k \), set the coefficients of the powers of \( k \) on both sides equal, and solve for the \( c \)’s.

Another way to do this, not pertinent to Gosper’s algorithm in general but pertinent to solving recursions, is to rewrite (1) as follows:
\[ f(k + 1) + f(k) = (E + 1)f(k) = (\Delta + 2)f(k) = 2(1 + \Delta/2)f(k) = -2(1 + \Delta/2)f(k) = -2(1 + \Delta/2)f(k) \]
so that a particular solution is given by
\[ f(k) = -\frac{1}{2} \frac{1}{1 + \Delta/2} (k + 2)^{N-1} = -\frac{1}{2} \sum_{r=0}^{\infty} \frac{\Delta}{r} (k + N)^{N-1} \frac{(N-1)\ldots(b-r+1)(b-r+1)}{(b-r+1)!} \]
Since \( \Delta \) decreases the degree of a nonzero polynomial by 1, the sum terminates at \( r = N - 1 \). By suitably modifying problem 5.1,
\[ \Delta^r(k + a)^b = b(b - 1)\ldots(b - r + 1)(b - r + 1)(b - r + a)^{b-r} = b^{b-r} \]
so
\[ f(k) = -\frac{1}{2} \sum_{r=0}^{N-1} \frac{\Delta}{r} (k + N)^{N-1} (N-1)\ldots(b-r+1)(b-r+1)! \]
Finally,
\[ s(k) = \frac{r(k)f(k - 1)}{p(k)} \]

\[ s(k) = \frac{f(k - 1)}{p(k)} \]

We are given \( a(k) = \frac{1}{k^r} \in \mathbb{Q}(k) \). If it has a hypergeometric antidifference \( s(k) \), then \( s(k) \) is a rational multiple of \( a(k) \) by Gosper’s algorithm, and hence \( s(k) \) is rational too; \( s(k) \in \mathbb{Q}(k) \).

The most general antidifference of \( a(k) \) is then \( s(k) + C \). For an arbitrary application of Gosper’s algorithm, \( s(k) + C \) would not be hypergeometric unless \( C = 0 \), but since \( s(k) \) is rational, so is \( s(k) + C \), so Gosper’s algorithm can return many possible functions, all differing by a constant.

By polynomial division, an arbitrary rational function \( s(k) \in \mathbb{Q}(k) \) can be written \( \gamma(k) + A0(k) / \beta(k) \) for polynomials \( \gamma(k), \alpha(k), \beta(k) \in \mathbb{Q}(k) \); \( A \in \mathbb{Q} \); and \( \alpha, \beta \) monic with \( \deg(\alpha) < \deg(\beta) \). Since we know \( s(k) \) approaches a limit as \( k \to \infty \), the polynomial \( \gamma(k) \) is just a
constant $C$: $s(k) = C + A\alpha(k)/\beta(k)$. All other possible antidifferences of $a(k)$ are obtained just by modifying $C$.

Then
\[
\sum_{k=1}^{n-1} a(k) = s(n) - s(1) = (C - C) + A\left(\frac{\alpha(n)}{\beta(n)} - \frac{\alpha(1)}{\beta(1)}\right)
\]

Since $\deg(\alpha) < \deg(\beta)$, $\lim_{n \to \infty} \frac{\alpha(n)}{\beta(n)} = 0$ so this gives
\[
\sum_{k=1}^{\infty} a(k) = -A\frac{\alpha(1)}{\beta(1)} \in \mathbb{Q}
\]

But here, the sum is known to be the irrational number $\frac{\pi^2}{6}$. There’s a contradiction; the assumption that a hypergeometric term antidifference $s(k)$ exists is false.

5.25

\[
s(k) = C + \sum_{n=0}^{k-1} q^{jn} = C + \frac{1 - q^{kj}}{1 - q^j}.
\]

For any fixed value $j = 1, 2, 3, \ldots$ and any constant $C$ (constant w.r.t. $k$; so $C \in \mathbb{Q}(q)$), this expression is a polynomial (over $\mathbb{Q}(q)$) in $q^k$ of degree $j$, so it’s $q$-hypergeometric. Gosper’s algorithm is shown on the worksheet.