Math 262a, Fall 1999, Glenn Tesler
Homework 4

> read 'hsum.mpl';

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**Problem 1**

First of all, the computer can generate the recurrence that is given to you in the problem:

```plaintext
> rec_cer := sumrecursion(2^k * n/(n-k) * binomial(n-k,2*k),
  k=0..n/3, f(n), certificate=true);
```

To recover what’s on the problem, replace \( s(n+i) \) by \( F(n+i,k) \) in the recurrence (first return value); and the second return value is \( R(n,k) \). Multiply it by \( F(n,k) \) to get \( G(n,k) \), then make the right side be \( G(n,k+1)-G(n,k) \).

```plaintext
> factor(-E^3+2*E^2-E+2);

so evidently the left side of the recursion has an overall minus sign from what’s given

```
so the right side has a negative also. So Koepf’s program deduced something equivalent to what’s stated. Now verify it.

**Problem 1a**

```maple
> lh := subs('f(n+i)=subs(n=n+i,F1)'$i=0..3, 
  lhs(rec_cer[1]));
lh := -2 \frac{2^k(n + 3) \text{binomial}(n + 3 - k, 2k)}{n + 3 - k} + 2 \frac{2^k(n + 2) \text{binomial}(n + 2 - k, 2k)}{n + 2 - k} 
- \frac{2^k(n + 1) \text{binomial}(n - k + 1, 2k)}{n - k + 1} + 2 \frac{2^k n \text{binomial}(n - k, 2k)}{n - k} 
> rh := subs(k=k+1,G1)-G1;
rh := 2(2^{k+1}) n \text{binomial}(n - k - 1, 2k + 2) (-n + k + 1) (2k + 1) (k + 1) 
\frac{(n - k - 1)(n + 3k + 2)(3k + 1 - n)(n + 3k)}{(n - k)(n + 3k - 1)(3k - 2 - n)(n + 3k - 3)} 
- 2 \frac{2^k n \text{binomial}(n - k, 2k) (-n + k)(2k - 1) k}{(n - k)(n + 3k - 1)(3k - 2 - n)(n + 3k - 3)} 
> simplify(lh/F1); 
\frac{25 k^3 - 25 n^2k^2 - 53 k^2 + 9 n^2k + 35 n k + 33 k - n^3 - 11 n - 6 - 6 n^2}{(n + 3k - 1)(3k - 2 - n)(n + 3k - 3)} 
> simplify(rh/F1); 
\frac{25 k^3 - 25 n^2k^2 - 53 k^2 + 9 n^2k + 35 n k + 33 k - n^3 - 11 n - 6 - 6 n^2}{(n + 3k - 1)(3k - 2 - n)(n + 3k - 3)} 
> "-";
0
```

The verification is complete.

**Problem 1b**

The bounds k=0..floor(n/3) aren’t natural: the denominator n-k cancels off the n-k from (n-k)! in the numerator, giving

```maple
> F1b := simpcomb(F1);
F1b := \frac{\Gamma(n - k) n 2^k}{\Gamma(2k + 1) \Gamma(n - 3k + 1)} 
```

and the k-support at n is k={0,1,...,floor(n/3), n,n+1,n+2,...}; for example,

```maple
> `'` F1`(10,kk)=limit(subs(n=10,F1b,k=kk)`$kk=-3..13;
```
but we do have \( F(n,k) = 0 \) for \( k = \lfloor n/3 \rfloor + 1, \lfloor n/3 \rfloor + 2, \ldots, n-1 \). From now on we must assume \( n \geq 2 \) because otherwise this \( k \)-range just given is empty, and we’re assuming it’s not.

The recursion involves \( f(n), f(n+1), f(n+2), f(n+3) \), and in all these the parameter is different, so the summation range is slightly different.

So take the relation
\[
-(En^2+1)(En-2) F1(n,k) = G1(n,k+1) - G1(n,k);
\]
and sum both sides from \( k=0..\lfloor n/3 \rfloor + 1 \). The terms on the left side still add up to \( \text{lhs(rec_cer[1])} \);

\[
-f(n+3) + 2f(n+2) - f(n+1) + 2f(n)
\]
because we have added 0 to some of these. The terms on the right side add up to \( G1(n, \lfloor k/3 \rfloor + 2) - G1(n,0) = 0 - 0 = 0 \). This proves the recursion above equals 0.

Now evaluate \( f(n) \). It’s a constant coefficient recurrence with roots \( i, -i, 2 \), so
\[
fn := c1*I^n + c2*(-I)^n + c3*2^n;
\]

\[
ff := nn \rightarrow \sum_{k=0}^{\lfloor n/3 \rfloor} \text{subs}(n=nn,F1), k=0..nn/3);
\]

Plug in initial conditions
\[
\text{subs(n=nn,fn) = ff(nn)'} \quad \text{nn=2..4};
\]

\[
-c1 - c2 + 4 c3 = 1, -I c1 + I c2 + 8 c3 = 4, c1 + c2 + 16 c3 = 9
\]

\[
solve({"}, \{c1,c2,c3\});
\]

\[
\{ c1 = \frac{1}{2}, c2 = \frac{1}{2}, c3 = \frac{1}{2} \}
\]

\[
\text{subs(",fn) ;}
\]

\[
\frac{1}{2} I^n + \frac{1}{2} (-I)^n + \frac{1}{2} 2^n
\]

\[
fn2 := ";
\]

Now check it.
\[
\text{for nn from 0 to 10 do}
\]

\[
\text{print('actual f'(nn)=ff(nn),}
\]

Page 3
new formula for $f'(nn) = \text{simplify}(\text{subs}(n=nn, fn2))$;

od;

    actual $f(0) = 0$, new formula for $f(0) = \frac{3}{2}$

    actual $f(1) = 1$, new formula for $f(1) = 1$

    actual $f(2) = 1$, new formula for $f(2) = 1$

    actual $f(3) = 4$, new formula for $f(3) = 4$

    actual $f(4) = 9$, new formula for $f(4) = 9$

    actual $f(5) = 16$, new formula for $f(5) = 16$

    actual $f(6) = 31$, new formula for $f(6) = 31$

    actual $f(7) = 64$, new formula for $f(7) = 64$

    actual $f(8) = 129$, new formula for $f(8) = 129$

    actual $f(9) = 256$, new formula for $f(9) = 256$

    actual $f(10) = 511$, new formula for $f(10) = 511$

We see that it happens to hold for $n=1$ by accident, and indeed does not hold for $n=0$.  

Problem 2

The examples shown:

> \text{sumrecursion}(\text{binomial}(n,k), k, s(n));

\[-s(n+1) + 2s(n) = 0\]

> \text{zeilberger}(\text{binomial}(n,k), k, s(n));

\[2s(n) - s(n+1) = 0\]

> \text{fasenmyer}(\text{binomial}(n,k), k, s(n), 0);

Error, (in kfreerec) no kfree recurrence equation of order (, 0, 0, ) exists

> \text{fasenmyer}(\text{binomial}(n,k), k, s(n), 1);

\[s(n+1) - 2s(n) = 0\]

> \text{closedform}(\text{binomial}(n,k), k, n);

$2^n$

> \text{Closedform}(\text{binomial}(n,k), k, n);

\[\text{Hyperterm}([1], [ ], 2, n)\]

and now doing the given problem with each of the various algorithms:

> \text{closedform}(\text{binomial}(n,k)^3, k, n);

Error, (in zeilberger) algorithm finds no recurrence equation of first order

> \text{recl} := \text{sumrecursion}(\text{binomial}(n,k)^3, k, s(n));
Creative telescoping found a recurrence of order 2. Try that for Celine’s algorithm.

They appear to be different!
But remember, any recursion can be "multiplied" by a polynomial in \( n \) and \( E \) to give a higher order recursion. So we could have \( r_2 = (A+B\cdot E)\cdot r_1 \), where \( A,B \) are polynomials in \( n \); or it could be the function really satisfies an order 1 recurrence, and \( r_1, r_2 \) are both multiples of it.

Set up the equation \( r_2 - (A+B\cdot E)\cdot r_1 = 0 \).
Collect it by \( s(n), s(n+1), \ldots \)
The coefficients of \( s(n), s(n+1), \ldots \) on both sides must agree, hence the coefficient on the left must equal 0. This gives a system of equations in \( A,B \).

The coefficients of \( s(n), s(n+1), \ldots \) on both sides must agree, hence the coefficient on the left must equal 0. This gives a system of equations in \( A,B \).
\[-(n + 3)^2 (3n + B + 4)\]

\[
> \text{solve}\{"\}, \{A, B\};
\]

\[
\{ B = -3n - 4, A = -3n - 7 \}
\]

\[
> \text{subs}("A+B*E);\]

\[-3n - 7 + (-3n - 4) \cdot E\]

Thus, the second recurrence is \[-((3n+7)+(3n+4)E)\cdot \text{first recurrence}.

SC is Sister Celine’s algorithm, CT is Zeilberger’s Creative Telescoping. Note SC succeeds implies CT succeeds (and with a possibly smaller order recurrence), but not conversely.

SC finds a homogeneous recurrence of unknown orders in n,k for F(n,k), then converts it to a recurrence for f(n)=sum_k F(n,k).

CT finds a nonhomogeneous recurrence for f(n) of unknown order in n alone, so there may not even be a recurrence for F(n,k), or it may exist, but with higher n-order than necessary for f(n).

Problem 3

Koepf 7.4

Let

\[
> \text{Sum}(a[i](n) \cdot s(n+i), i=0..K) = 0;
\]

\[
\text{Sum}(b[i](n) \cdot s(n+i), i=0..K) = 0;
\]

\[
\sum_{i=0}^{K} a_i(n) \cdot s(n+i) = 0
\]

\[
\sum_{i=0}^{K} b_i(n) \cdot s(n+i) = 0
\]

be two recurrence equations satisfied by s(n) of the same minimal order K. The first equation minus \(a_K(n)/b_K(n)\) * the second equation has order at most K-1, and thus is 0=0 because K was minimal. Thus, any two equations of minimal order are rational multiples of each other. Clearing denominators and removing all common factors, we can get the a_i(n)’s to be polynomials s.t. gcd(a_0(n),a_1(n),...,a_K(n))=1, and then all other recurrences of order K with polynomials in n as coefficients must be a polynomial multiple of this one.

Koepf 7.7
> sumrecursion(hyperterm([a,b],[c+m],1,k),k,s(m));
\[(b - c - m)(a - c - m)s(m + 1) + (c + m)(a + b - c - m) s(m) = 0\]

Koepf 7.10

> sumrecursion(binomial(n,k)*pochhammer(x,k)*pochhammer(y, n-k),k,s(n));
\[-s(n+1) + (y + x + n)s(n) = 0\]
> closedform(binomial(n,k)*pochhammer(x,k)*pochhammer(y, n-k),k,n);
pochhammer(y + x, n)

Koepf 7.11
Krawtchouk

> sumrecursion((-1)^n*p^n*binomial(N,n)*hyperterm([-n,-x],[ -N],1/p,k),k,K(n));
\[(2 + n)K(2 + n) - (2p - 1 - p N - n + 2np + x) K(1 + n) \\
+ p(-1 + p)(-N + n) K(n) = 0\]
> sumrecursion((-1)^n*p^n*binomial(N,n)*hyperterm([-n,-x], [-N],1/p,k),k,K(x));
p(x - N + 1)K(x + 2) - (2p - 1 - p N - x + 2xp + n) K(x + 1) \\
+ (x + 1)(-1 + p) K(x) = 0\]
> sumrecursion((-1)^n*p^n*binomial(N,n)*hyperterm([-n,-x], [-N],1/p,k),k,K(N));
\[-(1 + p)(-N - 2 + n) K(N + 2) - (x + 2p + n - 3 + p N - 2N) K(N + 1) \\
+ (x - N - 1) K(N) = 0\]

Koepf 7.15

> for dd from 2 to 5 do
print(`recursion for d`=dd);
rec := (sumrecursion((-1)^k*binomial(n,k)*binomial(dd*k,n),k,s(n)));
print(rec);
fn := subs(d_=dd, n -> (-d_)^n);
print(`Check at s`(n)=eval(fn),`
\`,expand(eval(subs(s=fn,rec))));
\end{aligned}
\[
\text{Koepf 7.19(d)}
\]

\[
> \text{closedform(binomial(-1/4, k)^2*binomial(-1/4, n-k)^2, k, n);} \\
\frac{((2n)!)^3}{(4^n)^3(n!)^6}
\]

\[
\text{Koepf 7.19(e)}
\]

This turned out to be much more tedious than I ever imagined it would be...

\[
> \text{Fnk := hyperterm([-n, 1-a-n, 1-b-n], [a, b], 1, k);} \\
\text{Fn := hypergeom([-n, 1-a-n, 1-b-n], [a, b], 1);} \\
\text{rec := sumrecursion(Fnk, k, s(n));}
\]

\[
Fnk := \frac{\text{pochhammer}(-n, k) \text{pochhammer}(1-a-n, k) \text{pochhammer}(1-b-n, k)}{\text{pochhammer}(a, k) \text{pochhammer}(b, k) k!}
\]
\[ Fn := \text{hypergeom}([-n, 1 - b - n, 1 - a - n], [a, b], 1) \]

\[ \text{rec} := (n + 2 b) (n + 2 a) (n + a + b) (n - 1 + a + b) s(n + 2) + (n + 1) (3 n + 2 + 2 a + 2 b) (2 a + 3 n + 2 b) (3 n + 2 b - 2 + 2 a) s(n) = 0 \]

It only involves \( s(n) \) and \( s(n+2) \), so it’s just as easy to solve as a first order recursion would be. But Maple doesn’t see that, so I aborted this computation after awhile.

\[ > \text{rsolve}(	ext{rec}, s(n)); \]

Warning, computation interrupted

Let’s transform it via \( n=2N \) (for even \( n \)), \( t(N)=s(2N) \). Then \( t(N+1)=s(2N+2) \) and it’s a first order recursion for \( t(N) \).

\[ > \text{s\_even := closedform}((n=2*N, \text{Fnk}), k, N); \]

\[
s\_even := (2 \, N)! \text{pochhammer}\left(\frac{1}{3} + \frac{1}{3} b + \frac{1}{3} a, N\right) \text{pochhammer}\left(\frac{1}{3} a + \frac{1}{3} b, N\right) \]

\[
\text{pochhammer}\left(-\frac{1}{3} + \frac{1}{3} b + \frac{1}{3} a, N\right) (-27)^N \bigg/ \left(4^N \text{pochhammer}(b, N) \right. \right.

\[
\text{pochhammer}(a, N) \text{pochhammer}\left(\frac{1}{2} a + \frac{1}{2} b, N\right) \text{pochhammer}\left(-\frac{1}{2} + \frac{1}{2} b + \frac{1}{2} a, N\right) N! \bigg) \right) \]

Using Koepf Exercise 1.3, this simplifies to the following (I found no built-in routine that will do this automatically; this was done by hand):

\[ > \text{s\_even2 := (-2)^((n/2)) * (n-1)! / (2^((n-2)/2) * ((n-2)/2)!) * pochhammer(a+b+n-1,n/2) / (pochhammer(a,n/2) * pochhammer(b,n/2));} \]

\[
s\_even2 := \frac{(-2)^{(1/2) n} (n - 1)! \text{pochhammer}\left(n - 1 + a + b, \frac{1}{2} n\right)}{2^{(1/2) n - 1} \left\{\frac{1}{2} n - 1\right\} \text{pochhammer}\left(a, \frac{1}{2} n\right) \text{pochhammer}\left(b, \frac{1}{2} n\right)} \]

Verify that \( s\_even, s\_even2, \) agree:

\[ '\text{simplify}((\text{subs}(N=\text{nn}, s\_even) - \text{subs}(n=2*\text{nn}, s\_even2)))' 'nn' = 1.5; \]

\[ 0, 0, 0, 0, 0 \]

Similarly for odd terms, do \( n=2N+1 \).

\[ > \text{s\_odd := closedform}((n=2*N+1, \text{Fnk}), k, N); \]

\[
s\_odd := \text{n!} \text{pochhammer}\left(\frac{1}{3} a + \frac{1}{2} + \frac{1}{3} b, N\right) \text{pochhammer}\left(\frac{1}{6} + \frac{1}{3} b + \frac{1}{3} a, N\right) \]

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\[ \text{pochhammer} \left( \frac{1}{3} a + \frac{5}{6} + \frac{1}{3} b, N \right) (-27)^N / \left( \text{pochhammer} \left( b + \frac{1}{2}, N \right) \right) \]
\[ \text{pochhammer} \left( a + \frac{1}{2}, N \right) \text{pochhammer} \left( \frac{1}{2} a + \frac{1}{2} b, N \right) \text{pochhammer} \left( \frac{1}{2} a + \frac{1}{2} b, N \right) \]

For odd \( n \), there is a catch --- this answer is absolutely wrong. It should be identically 0. There is a singularity of some sort that the program didn’t catch.

The series is terminating because the upper parameter \(-n\) results in \( \text{pochhammer}(-n,k)=0 \) for \( k \geq n \), so let’s compute it directly by summing \( F_{n,k}(n,k) \) over \( k=0..n \):

\[
\text{for } nn \text{ from 0 to 10 do } \text{FN}[nn] := \text{Fn}_\neg(nn) \od \]
\[
\text{FN}0 := \text{undefined} \\
\text{FN}1 := 0 \\
\text{FN}2 := -2 \frac{b + 1 + a}{a \ b} \\
\text{FN}3 := 0 \\
\text{FN}4 := 12 \frac{(a + b + 4) (a + b + 3)}{(1 + a) \ a \ b (1 + b)} \\
\text{FN}5 := 0 \\
\text{FN}6 := -120 \frac{(a + 7 + b) (a + 6 + b) (a + 5 + b)}{a (1 + a) (a + 2) \ b (1 + b) (b + 2)} \\
\text{FN}7 := 0 \\
\text{FN}8 := 1680 \frac{(a + 10 + b) (a + 9 + b) (a + 8 + b) (a + 7 + b)}{a (1 + a) (a + 2) (3 + a) \ b (1 + b) (b + 2) (3 + b)} \\
\text{FN}9 := 0 \\
\text{FN}10 := -30240 \frac{(a + 9 + b) (a + 13 + b) (a + 12 + b) (a + 11 + b) (a + 10 + b)}{(1 + a) (a + 2) (3 + a) (4 + a) (1 + b) (b + 2) (3 + b) (b + 4) b a}
\]

Empirically, when \( n \) is odd, the sum is 0. To prove it, we have the recursion above relating \( s(n) \) to \( s(n+2) \); since it’s 0 for \( n=1 \), it’s 0 for \( n=3,5,7,9,... \) by iterating the
recursion. Now we consider even \( n \).

\[
> \text{lcoeff(FN[10])}; \quad -30240
\]

\[
> \text{ifactor(lcoeff(FN[2*nn+2]) / lcoeff(FN[2*nn]))}$ nn' = 0..4
\]

\[-(2), -(2)(3), -(2)(5), -(2)(7), -(2)(3)^2\]

Empirically, when \( n = 2N \) is even, it is given by the formula above called \( s_{\text{even2}} \).

Now check whether this direct computation agrees with the automatically discovered formula \( s_{\text{even}} \):

\[
> \text{simplify(FN[2*nn] - subs(N=nn, s_{\text{even}}))}$ nn' = 1..5;
\]

\[
\begin{align*}
0, 0, 0, 0, 0
\end{align*}
\]

\[
> \text{Koepf 7.21}
\]

This is like the handout for Koepf problem 4.7 done in class 10/29/99. It’s almost the same except for the functions plugged in.

\[
> \text{sumrecursion(binomial(n-k,k)*x^k, k=0..n/2, s(n));}
\]

\[-s(n+2) + s(n+1) + x s(n) = 0\]

\[
> \text{sumrecursion(binomial(n+1,2*k+1)*(1+4*x)^k / 2^n, k=0..n/2, s(n));}
\]

\[-s(n+2) + s(n+1) + x s(n) = 0\]

The recursions are the same! Check initial conditions.

\[
> f1 := n \rightarrow \text{sum(binomial(n-k,k)*x^k, k=0..n/2)};
\]

\[
f1 := n \rightarrow \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^k
\]

\[
> f2 := n \rightarrow \text{sum(binomial(n+1,2*k+1)*(1+4*x)^k / 2^n, k=0..n/2)};
\]

\[
f2 := n \rightarrow \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\binom{n+1, 2k+1} (1+4x)^k}{2^n}
\]

\[
> f1(0) = f2(0), f1(1) = f2(1);
\]

\[
l = 1, 1 = 1
\]

That’s all that’s necessary. Check it for more. The variable \( nn \) was already set, so we must unset it.

\[
> \text{nn};
\]

\[
> \text{nn := 'nn'};
\]

\[
nn := nn
\]
Fibonacci numbers:
This sum at x=1 gives the Fibonacci numbers. Thus,

\[ \sum_{k=0}^{n} \binom{n-k}{k} x^k \]

Koepf 7.24(b)
We may write the product as

\[ \sum (\sum \text{Hyperterm}([a], [b], x, k) \times \text{Hyperterm}([a], [b], -x, m), k) \]
and letting \( n = m+k \), we pull out the factor \( x^n \) to get
\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \text{Hyperterm}(a, [b], 1, k) \text{Hyperterm}(a, [b], -1, 2n-k) \right) x^n
\]
Now find an expression for the inside summation.
\[
\text{sumrecursion}(\text{hyperterm}(a, [b], 1, k) \times \text{hyperterm}(a, [b], -1, 2n-k), k, n) = 0
\]
Once again, odd and even are different cases! We can tell from the definition of \( b \) that it is an even function of \( x \), so the odd terms all have coefficient 0. Thus, we rewrite the product again using only even exponents:
\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{2n} \text{Hyperterm}(a, [b], 1, k) \text{Hyperterm}(a, [b], -1, 2n-k) \right) x^{(2n)}
\]
That's better, now it will be able to solve it.
\[
\text{insidesum} := \text{closedform}(\text{hyperterm}(a, [b], 1, k) \times \text{hyperterm}(a, [b], -1, 2 \times n-k), k, n) \times x^{(2n)};
\]
\[
\text{insidesum} := \frac{\text{pochhammer}(a,n) \text{pochhammer}(-a+b,n) \left( \frac{1}{4} \right)^{n} x^{(2n)}}{\text{pochhammer} \left( \frac{1}{2} , n \right) \text{pochhammer} \left( b,n \right) \text{pochhammer} \left( \frac{1}{2} + \frac{1}{2} b, n \right) n!}
\]
\[
\text{Sumtohyper}(\text{insidesum}, n);
\]
\[
\text{Hypergeom} \left( a, -a+b, \left[ \frac{1}{2} - \frac{1}{2} b, b, b, \frac{1}{4} x^2 \right] \right)
\]