1–2. See Maple worksheet.

3. (a) Let \( D_1 = a_n^{2n-2} \prod_{i<j} (\alpha_i - \alpha_j)^2 \) and \( D_2 = \frac{(-1)^{n(n-1)/2}}{a_n} \) Res\((f, f', x)\).

   View \( D_1 \) and \( D_2 \) as polynomials in the roots \( \alpha_i \) of \( f \) and \( \beta_j \) of \( f' \). Let the degree of each \( \alpha_i \) and \( \beta_j \) be 1, and the degree of \( a_n \) be 0. Then \( D_1 \) has degree \( 2\binom{n}{2} = n(n-1) \) because it’s a product indexed by pairs \( 1 \leq i < j \leq n \), each factor having degree 2. The resultant \( \text{Res}(f, f', x) \) is the product of \( n(n-1) \) factors \( \alpha_i - \beta_j \), each with degree 1. So \( D_1 \) and \( D_2 \) have the same degree.

   The product \( \prod_{i<j} (\alpha_i - \alpha_j)^2 \) is a symmetric function of the \( \alpha \)'s, homogeneous of degree \( 2\binom{n}{2} = n(n-1) \), so it may be expressed as a polynomial in \( e_k(\alpha_1, \ldots, \alpha_n) = -a_{n-k}/a_n \) for \( k = 1, \ldots, n \). Thus, \( D_1 \) is a polynomial in the \( a_k \)'s divided by some power of \( a_n \), and it vanishes iff \( f(x) \) has a repeated root.

   The second definition using the resultant also is a polynomial in the \( a_k \)'s, divided by \( a_n \); but the bottom row of the Sylvester matrix is divisible by \( a_n \), leaving just a polynomial in \( a_n \). The resultant vanishes iff \( f(x) \) and \( f'(x) \) have a common root iff \( f(x) \) has a repeated root, so the same applies to \( D_2 \).

   Since \( D_1 \) and \( D_2 \) have the same degrees and vanishing conditions, \( D_2 = p \cdot D_1 \) for a function \( p \) of the \( a \)'s that never vanishes (provided \( a_n \neq 0 \)). So \( p = C a_n^k \) for suitable constants \( C \) and \( k \). Suppose \( \hat{f} \) is monic and \( f = a_n \hat{f} \). Let \( \hat{D}_1 \) and \( \hat{D}_2 \) be the \( D_1 \) and \( D_2 \) for \( \hat{f} \). Then \( \text{Res}(f, f', x) = a_n^{2n-1} \text{Res}(\hat{f}, \hat{f}', x) \) and \( D_2 = a_n^{2n-2} D_2 \), and we see that \( D_1 \) scales this way too. So we chose the correct exponent, and \( D_2 = C D_1 \) for a constant \( C \), which turns out to be as above. [INCOMPLETE]

(b) The resultant is

\[
\text{Res}(ax^2 + bx + c, 2ax + b, x) = \det \begin{bmatrix} 2a & 0 & a \\ b & 2a & b \\ 0 & b & c \end{bmatrix} = -a(b^2 - 4ac)
\]

so the discriminant is \( b^2 - 4ac \).

4. (a) In Koepf # 5.1 it was shown that for \( n^k = \binom{n}{k} \), \( \Delta n^k = k \cdot n^{k-1} \).

   Iterating, \( \Delta^r n^k = k^r \cdot n^{k-r} \). Note that \( n^r \biggr|_{n=0} = 0 \) when \( r > n \). Thus

\[
\Delta^r n^k \biggr|_{n=0} = \begin{cases} 0 & \text{if } r \neq k; \\ k! & \text{if } r = k. \end{cases}
\]

   The functions \( \{ n^0, \ldots, n^D \} \) span all polynomials of degrees \( \leq D \), so we may represent any polynomial in the form

\[
f(n) = \sum_{k=0}^{\infty} b_k n^k
\]

for suitable constants \( b_k \), only finitely many of which are nonzero. Apply \( \Delta^r \) and set \( n = 0 \) to get \( (\Delta^r f)(0) = k! b_k \). Then a more convenient way to write \( f(n) \) is

\[
f(n) = \sum_{k=0}^{\infty} a_k \frac{n^k}{k!} \quad \text{where} \quad a_k = (\Delta^k f)(0).
\]
To compute the given sum, let \( f(n) = 5n^3 + 4n^2 \). We have

\[
\begin{array}{c|c|c}
  k & \Delta^k f(n) & (\Delta^k f)(0) \\
  \hline
  0 & 5n^3 + 4n^2 & 0 \\
  1 & 15n^2 + 23n + 9 & 9 \\
  2 & 30n + 38 & 38 \\
  3 & 30 & 30 \\
  k \geq 4 & 0 & 0 \\
\end{array}
\]

so

\[
f(n) = \frac{9}{1!} n^1 + \frac{38}{2!} n^2 + \frac{30}{3!} n^3.
\]

and an antidifference \( F(n) \) s.t. \( \Delta F(n) = f(n) \) is

\[
F(n) = \frac{9}{2!} n^2 + \frac{38}{3!} n^3 + \frac{30}{4!} n^4.
\]

Thus

\[
\sum_{n=0}^{m} 5n^3 + 4n^2 = F(m + 1) - F(0)
\]

\[
= \frac{5}{4} (m + 1)^4 - \frac{7}{6} (m + 1)^3 - \frac{3}{4} (m + 1)^2 + \frac{2}{3} (m + 1)
\]

\[
= \frac{5}{4} m^4 + \frac{23}{6} m^3 + \frac{13}{4} m^2 + \frac{2}{3} m
\]

(b) Let \( b_k(n) = \Gamma(n)/\Gamma(n + k + \alpha) \).

First evaluate \( \Delta \) and \( n \) applied to this basis:

\[
\Delta b_k(n) = b_k(n) \left( \frac{n}{n + k + \alpha} - 1 \right) = b_k(n) \left( 1 - \frac{k + \alpha}{n + k + \alpha} - 1 \right) = -(k + \alpha) b_{k+1}(n)
\]

\[
n b_k(n) = \frac{n}{n + k + \alpha - 1} b_{k-1}(n) = b_{k-1}(n) - (k + \alpha - 1) b_k(n)
\]

Combine these to get

\[
n \Delta b_k(n) = -(k + \alpha) n b_{k+1}(n)
\]

\[
= -(k + \alpha) (b_k(n) - (k + 1 + \alpha - 1) b_{k+1}(n))
\]

\[
= -(k + \alpha) b_k(n) + (k + \alpha)^2 b_{k+1}(n)
\]

Set

\[
f(n) = \sum_{k=0}^{\infty} c_k b_k(n)
\]

and plug this into \( n \Delta f(n) = f(n) \) to obtain

\[
\sum_{k=0}^{\infty} c_k \left( -(k + \alpha) b_k(n) + (k + \alpha)^2 b_{k+1}(n) \right) = \sum_{k=0}^{\infty} c_k b_k(n)
\]

Collecting coefficients in terms of this basis gives

\[
0 = \sum_k \left( -(k + \alpha)c_k + (k - 1 + \alpha)^2 c_{k-1} - c_k \right) b_k(n) = \sum_k \left( -(k + \alpha + 1)c_k + (k + \alpha - 1)^2 c_{k-1} \right) b_k(n)
\]

where we sum over all \( k \) with the understanding that \( c_k = 0 \) for \( k \leq 0 \). This gives the recursion

\[
-(k + \alpha + 1)c_k + (k + \alpha - 1)^2 c_{k-1} = 0.
\]
Now get the indicial equation: at $k = 0$ this becomes
\[ 0 = -(0 + \alpha + 1)c_0 + (0 + \alpha - 1)^2c_{-1} = -(\alpha + 1)c_0 + (\alpha - 1)^2 \cdot 0 = -(\alpha + 1)c_0 \]
where $c_{-1} = 0$ and $c_0 \neq 0$; thus, $\alpha = -1$ is the root of the indicial equation. So the final recursion for the $c$'s is
\[ -k c_k + (k - 2)^2c_{k-1} = 0 \quad \text{so} \quad c_k = \frac{(k - 2)^2}{k}c_{k-1} \quad \text{for} \quad k \geq 1. \]
Iterating gives $c_1 = c_0$, $c_2 = c_3 = \cdots = 0$. So the final answer is
\[ f(n) = c_0(b_0(n) + b_1(n)) = c_0 \left( \frac{\Gamma(n)}{\Gamma(n - 1 + 0)} + \frac{\Gamma(n)}{\Gamma(n - 1 + 1)} \right) = c_0 ((n - 1) + (1)) = \boxed{c_0 n}. \]
Alternately, rewrite the original equation as
\[ n(f(n + 1) - f(n)) = f(n) \quad \Rightarrow f(n + 1) = \frac{n + 1}{n}f(n) \quad \Rightarrow \boxed{f(n) = n f(1)}. \]