Homework #1

1. Consider an approximation of the form

$$\int_a^b f(x) \, dx \approx \sum_{i=0}^{n} A_i f(x_i),$$

for some constants $A_i$ and locations $x_i$. If this approximation is exact for $f(x) = x^j$, $j = 0, \ldots, r$, show the approximation is exact for any $f \in \Pi_r$.

2. (a) Verify the approximation

$$\int_{-1}^{1} f(x) \, dx \approx f(-\sqrt{3}/3) + f(\sqrt{3}/3)$$

is exact for $f(x) = x^j$, $j = 0, \ldots, 3$.

(b) Find the approximation and absolute error when $f(x) = x^4$.

3. Consider the approximation formula

$$\int_{-1}^{1} f(x) \, dx \approx Af \left(-\sqrt{\frac{3}{5}}\right) + Bf(0) + Cf \left(\sqrt{\frac{3}{5}}\right).$$

(a) Find $A, B, C$ such that the approximation is exact for $f \in \Pi_2$.

(b) How do we change this formula for application to $\int_a^b f(x) \, dx$?

(c) Find the approximation and absolute error when $a = 0, b = \pi, f(x) = \sin x$.

4. Consider the weight function $w(x) = x$.

(a) Find the monic degree 2 polynomial that is $w$-orthogonal to $\Pi_1$ in $[1, 2]$.

(b) Derive the Gaussian quadrature formula for approximating $\int_1^2 f(x)w(x) \, dx$ using 2 nodes.

(c) Find the approximation and absolute error when $f(x) = e^x$.

5. Suppose we are interested in using Gaussian quadrature $(w \equiv 1)$ for approximating

$$\int_0^{1/2} \cos x \, dx$$

by an absolute error $\leq \epsilon$, for some given tolerance $\epsilon > 0$.

(a) Show, using the error term for Gaussian quadrature, that we can do this by using $m$ nodes, where $m$ satisfies

$$\frac{1}{(2m)!} \frac{1}{2^{2m+1}} < \epsilon.$$
(b) Plug in some $m$ into this expression to find how many nodes will allow us to have an absolute error $\leq 10^{-4}$.

6. Prove no approximation formula of the form
\[
\int_a^b f(x) \, dx \approx \sum_{i=0}^n A_i f(x_i)
\]
can be exact for all $f \in \Pi_{2n+2}$.

7. A function $F : \mathbb{R} \to \mathbb{R}$ is called even if $F(x) = F(-x)$, and is called odd if $F(x) = -F(-x)$. Consider the interval $[-c, c]$, for some $c > 0$, and an even, positive weight function $w$ defined on $[-c, c]$.

(a) Prove if $q$ is a degree $n \geq 1$ polynomial that is $w$-orthogonal to $\Pi_{n-1}$ on $[-c, c]$, then $q$ is even when $n$ is even and $q$ is odd when $n$ is odd.

(b) For the roots of this $q$, show $x$ is a root implies $-x$ is also a root. Also show 0 is a root in the case $q$ is odd. What do these say about Gaussian quadrature formulas for $\int_{-c}^{c} f(x)w(x) \, dx$?

8. (Matlab) Use only basic programming in the following. First write a function that defines a weight function. For example,

function [y] = weightfn(x)
    y = 1+x*x; end

Then write a function that does the following:

- Input: number of nodes, $m$; node locations, $x_i$, given in an array; left interval endpoint, $a$; right interval endpoint, $b$; specific node number, $j$; number of evenly sized subintervals for composite Simpson’s rule, $n$, even.
- Output: the coefficient $A_j$ for the approximation

\[
\int_a^b f(x)w(x) \, dx \approx \sum_{i=1}^m A_i f(x_i)
\]

that is exact for $f \in \Pi_m$ using composite Simpson’s rule for the integration.

For example, your header for this function may look like

function [A] = getcoef(m,x,a,b,j,n)
    :
end

(a) Write out or print out your program.

(b) Apply your program to verify your coefficients in 3a. Write out or print out (using the “diary” command) your work.

(c) Apply your program to the case $a = -1, b = 1, x = -1:0.1:1, n = 200$ and $w \equiv 1$ to find the coefficient $A_4$. Write out or print out your work.