Homework #4

1. Consider the initial value problem

\[
\begin{cases}
    x' = 1 + x/t, & 1 \leq t \leq 2 \\
    x(1) = 2
\end{cases}
\]

with exact solution \( x(t) = t \ln t + 2t \). Find the absolute errors, \( E_h \), at \( t = 2 \) for modified Euler’s method with the two step sizes \( h = 1/2 \) and \( 1/4 \). Does the ratio \( E_{1/2}/E_{1/4} \) indicate higher order accuracy?

2. Consider the initial value problem

\[
\begin{cases}
    x' = \sin (\pi t) \sin x, & 0 \leq t \leq 1 \\
    x(0) = 1
\end{cases}
\]

(a) Find the approximation of modified Euler’s method at \( t = 0.2 \) using stepsize \( h = 0.1 \).

(b) Find the approximation of Heun’s method at \( t = 0.2 \) using stepsize \( h = 0.1 \).

3. Consider the initial value problem

\[
\begin{cases}
    x' = x + t, & 1 \leq t \leq 2 \\
    x(1) = -1
\end{cases}
\]

(a) Find the approximations at \( t = 3/2, 2 \) of Runge-Kutta of order 4 using stepsize \( h = 1/2 \).

(b) Make a list of the locations \( f(t, x) = x + t \) needed to be evaluated at in 3a.

4. Consider the initial value problem

\[
\begin{cases}
    x' = x + t, & 1 \leq t \leq 2 \\
    x(1) = -1
\end{cases}
\]

(a) Find the stepsize to use in the first step of adaptive Runge-Kutta-Fehlberg for a local truncation error tolerance of \( 10^{-4} \) using the initial guess \( h = 0.1 \).

(b) At the end of the first step, what stepsize do we have to try for the second step (double or not)? What, instead, is the value of the guess \( 0.9h[\delta/|e|]^{1/(1+p)} \)?

5. (Matlab) Consider the initial value problem

\[
\begin{cases}
    x' = f(t, x), & a \leq t \leq b \\
    x(a) = \alpha
\end{cases}
\]

First write a function for \( f \). Then write a function that does the following:
• Input: $a, b, \alpha$, and stepsize $h$.
• Output: Approximations of the $x(t_i)$ using Runge-Kutta of order 4.

(a) Write out or print out your program.
(b) For $f(t, x) = x + t$ with $a = 1, b = 2, \alpha = -1$, plot your approximations for $h = 0.01$. Also write or print out the approximation at $x(2)$.
(c) For $f(t, x) = x + t$ with $a = 1, b = 2, \alpha = -1$, an using the exact solution, find the absolute errors $E_h$ at $t = b$ for stepsizes $h = 0.01, 0.005$. Guess the order of accuracy of the method from $E_{0.01}/E_{0.005}$. Write or print out your work.