Homework #6

1. Show the Adams-Moulton methods are all stable.

2. Consider the method
   \[ x_{n+1} = x_n + \frac{h}{2} [f_{n+1} + f_n] \]
   applied to the initial value problem
   \[
   \begin{cases}
   x' = f(t, x), & a \leq t \leq b \\
   x(a) = \alpha
   \end{cases}
   \]
   (a) Show the method is consistent.
   (b) Let \( x(t) \) be the exact solution of the initial value problem. Show
   \[
   x(t_{n+1}) - x(t_n) - \frac{h}{2} [f(t_{n+1}, x(t_{n+1})) + f(t_n, x(t_n))] = O(h^m)
   \]
   for some \( m \).
   (c) Thus, what is the order of the local truncation error for the multistep method? Verify this order is \( \geq 2 \), which is required for consistency.

3. Determine if the multistep method
   \[ x_{n+1} = x_{n-1} + 2hf_n \]
   is convergent.

4. Discuss the stability and consistency of the multistep methods:
   (a) \[ x_{n+1} - x_n = h \left[ \frac{3}{5} f_{n+1} + \frac{19}{24} f_n - \frac{5}{24} f_{n-1} + \frac{1}{24} f_{n-2} \right] \]
   (b) \[ x_{n+1} + 4x_n - 5x_{n-1} = h [4f_n + 2f_{n-1}] \]
   (c) \[ x_{n+1} - 3x_n + 2x_{n-1} = h [f_{n+1} + 2f_n + f_{n-1} - 2f_{n-2}] \]

5. A multistep method is called weakly stable if \( p \) has a zero \( \lambda \) satisfying: \( \lambda \neq 1, |\lambda| = 1 \), and \( q(\lambda) < \lambda p'(\lambda) \). Show Milne’s method,
   \[ x_{n+1} - x_{n-1} = h \left[ \frac{1}{3} f_{n+1} + \frac{4}{3} f_n + \frac{1}{3} f_{n-1} \right], \]
   is weakly stable.

6. A multistep method is called strongly stable if \( p(1) = 0, p'(1) \neq 0 \), and all other roots \( z \) of \( p \) satisfy \( |z| < 1 \). Which of the methods in problem 4 are strongly stable?
7. (Matlab) Consider the initial value problem
\[
\begin{align*}
\dot{x} &= 0, \quad t \geq 0 \\
x(0) &= 1
\end{align*}
\]
Program up the multistep method
\[x_{n+1} + 4x_n - 5x_{n-1} = h[4f_n + 2f_{n-1}]\]
applied to the initial value problem satisfying:

- Input: stepsize \( h \); initial guess at 0, \( x_0 \); initial guess at \( h \), \( x_1 \); \( N \).
- Output: \( x_i \), for \( i = 0, \ldots, N \).

(a) Write out or print out your program.
(b) Run your program with \( h = 0.01 \), \( x_0 = 1 \), \( x_1 = 1 \), \( N = 100 \) and print out a plot of the \( x_i \). Are these approximate values close to the exact solution, \( x(t) = 1 \)?
(c) Run your program with \( h = 0.01 \), \( x_0 = 1 \), \( x_1 = 1.0001 \), \( N = 10 \) and print out a plot of the \( x_i \). Also look at what happens with larger \( N \). Are these approximate values close to \( x(t) = 1 \)?