Adaptive Simpson’s Rule Example

Consider the integral
\[ \int_1^3 e^{2x} \sin(3x) \, dx, \]
and the error tolerance \( \epsilon = 0.2 \). We apply a few steps of the adaptive simpson’s rule method.

Let 
\[ f(x) = e^{2x} \sin(3x), \]
the integrand.

**Step 1:** Approximation at this step is \( S(1, 2) + S(2, 3) \). We first check the error of this. This is through the formula
\[
\frac{1}{10} \left| S(a, b) - S(a, \frac{a+b}{2}) - S(\frac{a+b}{2}, b) \right|. 
\]

Note we use \( \frac{1}{10} \) rather than \( \frac{1}{15} \) to be safe. So calculating,
\[
S(1, 3) = 35.42697658812284 \\
S(1, 2) = -15.45828245392933 \\
S(2, 3) = 117.9751755250024,
\]
and thus our approximation of error is
\[
|f(x) - S(1, 2) - S(2, 3)| \approx 6.70899164829503.
\]

This error is not yet acceptable since it is greater than \( \epsilon = 0.2 \), so we add midpoints and continue to the next step.

**Step 2:** Approximation at this step is \( S(1, 1.5) + S(1.5, 2) \) for the integral in \([1, 2]\) and \( S(2, 2.5) + S(2.5, 3) \) for the integral in \([2, 3]\). Thus the total approximation is the sum of these for the integral in \([1, 3]\). Calculating,
\[
S(1, 2) = -15.45828245392933 \\
S(1, 1.5) = -3.87030357255464 \\
S(1.5, 2) = -12.38881686458909 \\
\]
and

\[
S(1, 3) = 117.9751755250024 \\
S(1, 2) = 23.83355636842984 \\
S(2, 3) = 100.7072692285579,
\]

leading to the error approximations of 0.08008379832144 for the integral in [1, 2] and −0.65656500719853 for the integral in [2, 3]. This error in [1, 2] is acceptable since it is less than \( \frac{\epsilon}{2} = 0.1 \). The error in [2, 3], however, is not and we add midpoints and continue with it to the next step.

**Step 3:** Approximation at this step is \( S(2, 2.25) + S(2.25, 2.5) \) for the integral in [2, 2.5] and \( S(2.5, 2.75) + S(2.75, 3) \) for the integral in [2.5, 3]. Calculating,

\[
S(1, 2) = 23.83355636842984 \\
S(1, 1.5) = 2.12361566688147 \\
S(1.5, 2) = 21.85661747203629
\]

and

\[
S(1, 2) = 100.7072692285579 \\
S(1, 1.5) = 46.96091888208836 \\
S(1.5, 2) = 53.89025695750476,
\]

leading to the error approximations of 0.01466767704879 for the integral in [2, 2.5] and 0.01439066110352 for the integral in [2.5, 3]. Both errors are acceptable since they are less than \( \frac{\epsilon}{4} = 0.5 \). Thus we stop at this step.

The final approximation for our problem is the sum

\[
S(1, 1.5) + S(1.5, 2) + S(2, 2.25) + S(2.25, 2.5) + S(2.5, 2.75) + S(2.75, 3).
\]

The first two terms sum up to \(-16.25912043714373\), which approximates the integral in [1, 2]. The next two sum up to 23.98023313891776, which approximates the integral in [2, 2.5]. The last two sum up to 100.8511758395931, which approximates the integral in [2.5, 3]. Thus, in total, the sum 108.5722885413671 approximates the whole integral with an approximate error less than \( \epsilon = 0.2 \).

If the error tolerance is small, this process should be programmed up.