Homework #2

1. (a) Give a graphical description showing how fixed point iterations converge for the fixed point function \( g(x) = x/2 \).

(b) Give a graphical description showing how fixed point iterations do not converge for the fixed point function \( g(x) = 2x \).

2. Let \( g(x) = 1/x + x/2 \) and consider the interval \([1.4, 1.45]\).

(a) Show \( g(x) \in [1.4, 1.45] \) for \( x \in [1.4, 1.45] \) by finding the maximum and minimum values of \( g \) in the interval.

(b) Find \( 0 \leq \lambda < 1 \) such that \(|g'(x)| \leq \lambda \) for all \( x \in [1.4, 1.45] \) by finding the maximum and minimum values of \( g'(x) \) in the interval.

(c) Use this \( \lambda \), along with error bounds, to estimate \( n \) such that \( p_n \) of fixed point iterations will have absolute error \( \leq 10^{-4} \), when \( p_0 = 1.425 \). Do the same for absolute error \( \leq 10^{-10} \).

(d) Perform fixed point iterations with initial guess \( p_0 = 1.425 \) until \(|p_k - p_{k-1}| \leq 10^{-4}\) is satisfied.

3. Consider the root-finding problem \( x^2 - 3 = 0 \).

(a) Consider \( x^2 + x - 3 = x \), obtained by adding \( x \) on both sides. Study the value of \(|g'(\sqrt{3})|\) and comment on whether fixed point iterations will converge.

(b) Find a different way of turning \( x^2 - 3 = 0 \) into a fixed point problem that gives a fixed point function \( g(x) \) that satisfies \(|g'(\sqrt{3})| < 1\). Comment on the convergence of the fixed point iterations.

4. (a) Starting with initial guess \( p_0 = 1 \), find approximations \( p_1, p_2, p_3 \) to the root of \( f(x) = x^2 - 3 \) using Newton’s method.

(b) Give a graphical description of how Newton’s method arrives at these approximations.

5. (a) Consider

\[
 f(x) = \begin{cases} 
 \sqrt{x}, & x \geq 0 \\
 -\sqrt{-x}, & x < 0. 
\end{cases}
\]

Starting with initial guess \( p_0 = a > 0 \), find 3 additional approximations to the root of \( f(x) \) using Newton’s method.

(b) Give a graphical description of how Newton’s method arrives at these approximations.

(c) Will Newton’s method converge to the exact root at 0 for any \( p_0 \neq 0 \)? Why does this not violate the theorem on convergence of Newton’s method?
6. (a) Starting with the initial guess $p_0 = 1.9$, find approximations $p_1, p_2, p_3$ to the root of $f(x) = (x - 2)^2$ using Newton's method.

(b) Let $e_n = |2 - p_n|$, for $n = 0, 1, 2, 3$. Calculate $e_n$. Also calculate $e_{n+1}/e_n$ and $e_{n+1}/e_n^2$, for $n = 0, 1, 2$. Does the sequence of approximations look to have order of convergence 1? What about 2?

(c) Starting with initial guess $p_0 = a$, find one additional approximation to the root of $f(x) = (x - 2)^m$, for $m$ a positive integer, using the iterations

$$p_{n+1} = p_n - m \frac{f(p_n)}{f'(p_n)}.$$

7. (Matlab)

(a) Write a Matlab function that inputs a number $x$ and outputs the value $\cos x$. Then write a Matlab function that inputs a starting guess $p_0$ and tolerance $\epsilon$ and outputs the number of iterations $n$ and the final fixed point approximation $p_n$ satisfying $|p_n - p_{n-1}| \leq \epsilon$ for the fixed point function $g(x) = \cos x$. Print out or write out this function.

(b) Run your function using $p_0 = 1$ and $\epsilon = 10^{-10}$ and print out or write out the results.

8. (Math 274)

(a) Suppose $g$ and $g'$ are continuous functions. Prove if $p$ is a fixed point of $g$ and $|g'(p)| < 1$, then fixed point iterations will converge if $p_0$ is close enough to $p$ (use continuity of $g'$ to show there is an interval $[p - \delta, p + \delta]$ where $|g'(x)| \leq \lambda < 1$ and then try to use the theorem on convergence).

(b) Suppose $g$ and $g'$ are continuous functions. Prove if $|g'(x)| \geq 1$ everywhere, then fixed point iterations will not converge to any fixed point when the starting guess is not itself a fixed point.