Fifth(a) Homework

Question 1:
Gaussian quadrature also works on weighted integrals, with given weight \( w(x) \),
\[
\int_a^b w(x)f(x) \, dx.
\]
In this case, for a given number of nodes \( n + 1 \), the formula
\[
\sum_{j=0}^{n} a_j f(x_j)
\]
chooses node locations \( x_j, j = 0, \ldots, n \) at the roots of the degree \( n + 1 \) orthogonal polynomial \( \phi_{n+1} \) that works over \([a, b]\) with respect to the weight \( w(x) \). Furthermore, the coefficients satisfy
\[
a_j = -\frac{A_{n+2}}{A_{n+1}} \frac{1}{\phi'_{n+1}(x_j)\phi_{n+2}(x_j)} \int_a^b w(x)(\phi_{n+1}(x))^2 \, dx,
\]
where \( A_j \) is the coefficient of \( x^j \) for \( \phi_j \). This formula has degree of precision \( 2n + 1 \), when considering \( f(x) \) equal to polynomials. Write down the degree of precision 3 version when the interval is \([-1, 1]\) and the weight is \( w(x) = \frac{1}{\sqrt{1-x^2}} \).

Question 2:
Suppose \( N(h) \) is an approximation of \( M \) with the finite expansion
\[
M = N(h) + 2h - \frac{3}{2} h^2 + O(h^3).
\]
Apply the Richardson extrapolation idea twice using \( N(h) \), \( N(\frac{h}{3}) \), and \( N(\frac{h}{9}) \) to get an approximation of \( M \) of order \( O(h^3) \).

Question 3:
Romberg integration leads to Composite Simpson’s Rule after the first iteration of Richardson’s extrapolation. Write down the approximation formula coming out of the second iteration.

Question 4:
The Euler-MacLaurin formula gives an expansion of
\[
M = N(h) + K_2 h^2 + K_4 h^4 + \ldots,
\]
where
\[ K_{2i} = -\frac{B_{2i}}{(2i)!} (f^{(2i-1)}(b) - f^{(2i-1)}(a)), \]
for the Composite Trapezoidal Rule approximation \( N(h) \) to the exact integral \( M = \int_a^b f(x) \, dx \).

Using Richardson’s extrapolation, determine a similar expansion for Composite Simpson’s rule.

**Question 5:**
Perform 3 steps of Romberg integration to get \( N_3(1) \) for the integral
\[ \int_0^2 x^3 \, dx. \]

Why is the approximation exact after the second step?