Problem 1 (WR Ch 5 #11). Suppose \( f \) is defined in a neighborhood of \( x \), and suppose \( f''(x) \) exists. Show that
\[
\lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2} = f''(x).
\]
Show by an example that the limit may exist even if \( f''(x) \) does not.

Solution. Since \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \), and since \( h \to 0 \iff -h \to 0 \), if we replace \( h \) by \(-h\) in this expression, we have
\[
f'(x) = \lim_{h \to 0} \frac{f(x + (-h)) - f(x)}{(-h)} = \lim_{h \to 0} \frac{f(x) - f'(x - h)}{h}.
\]
Therefore,
\[
f''(x) = \lim_{h \to 0} \frac{f(x + h) - f'(x)}{h} = \lim_{h_1 \to 0} \frac{f(x + h) - f(x + h_1)}{h_1} - \lim_{h_2 \to 0} \frac{f(x) - f(x - h_2)}{h_2},
\]
and letting \( h_1 = h = h_2 \), i.e. taking them all to zero at the same rate (which we can do by Theorem 4.2), we have
\[
f''(x) = \lim_{h \to 0} \frac{f(x + h) - f(x - h)}{h} = \lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2}.
\]
For the second portion, we would like to find an \( f \) so that \( f'(x) = |x| \). One such choice of \( f \) could be \( f(x) = \int_0^x |t| \, dt = \frac{1}{2} x^2 \). Now let \( x = 0 \). By continuity of \( f \) at \( x = 0 \) we have
\[
\lim_{h \to 0} [f(x + h) + f(x - h) - 2f(x)] = f(0) + f(0) - 2f(0) = 0.
\]
Also, clearly \( \lim_{h \to 0} h^2 = 0 \). So by L'Hôpital's rule we have
\[
\lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2} \overset{L'H}{=} \lim_{h \to 0} \frac{f'(x + h) - f'(x - h)}{2h} = \lim_{h \to 0} \frac{|0 + h| - |0 - h|}{2h} = 0.
\]
That means the limit exists at \( x = 0 \), but \( f''(x) = |x| \) is not differentiable at 0, so \( f''(0) \) does not exist.

Problem 2 (WR Ch 5 #12). If \( f(x) = |x|^3 \), compute \( f'(x) \), \( f''(x) \) for all real \( x \), and show that \( f^{(3)}(0) \) does not exist.
Solution. For $x \neq 0$, $|x|$ is a differentiable function with derivative
\[
\text{sgn}(x) = \begin{cases} 
1 & \text{if } x > 0 \\
-1 & \text{if } x < 0
\end{cases}.
\]
Thus by the chain rule in the first line and by the product rule in the second line,
\[
f'(x) = 3|x|^2 \text{sgn}(x) = 3x|x|.
\]
\[
f''(x) = 3|x| + 3x \text{sgn}(x) = 3|x| + 3|x| = 6|x|.
\]
Checking the cases for $x = 0$ by hand, we have
\[
f'(0) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{|h|^3 - 0}{h} = \lim_{h \to 0} h| |h| = 0.
\]
\[
f''(0) = \lim_{h \to 0} \frac{f'(x + h) - f'(x)}{h} = \lim_{h \to 0} \frac{3h| |h| - 0}{h} = \lim_{h \to 0} 3|h| = 0.
\]
\[
f'''(0) = \lim_{h \to 0} \frac{f''(x + h) - f''(x)}{h} = \lim_{h \to 0} \frac{6|h| - 0}{h} = \lim_{h \to 0} 6 \frac{|h|}{h} = \text{DNE}.
\]

Problem 3 (WR Ch 6 #2). Suppose $f \geq 0$, $f$ is continuous on $[a, b]$, and $\int_a^b f(x) \, dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution. Assume by way of contradiction that there is some $y \in [a, b]$ such that $f(y) > 0$, and let $\epsilon = \frac{f(y)}{2}$. Since $f$ is continuous, there exists a $\delta > 0$ such that
\[
0 < |y - x| < \delta \quad \implies \quad |f(y) - f(x)| < \epsilon = \frac{f(y)}{2},
\]
for $x \in [a, b]$. This last inequality gives us
\[
f(y) - f(x) \leq |f(y) - f(x)| < \frac{f(y)}{2} \implies f(x) > \frac{f(y)}{2} > 0.
\]
Let $I = (y - \delta, y + \delta) \cap [a, b]$. What we have shown so far is that if $x \in I$, then $f(x) > \frac{f(y)}{2} > 0$.

Now, given some partition $P$ of $[a, b]$, we make a refinement $P^*$ by adding if necessary (and if possible) a point in $(y - \delta, y) \cap [a, b]$ and a point in $(y, y + \delta) \cap [a, b]$ so that $y \in (x_{k-1}, x_k) \subset I$ with $x_{k-1}, x_k \in P^*$ for some $1 \leq k \leq n$. Then
\[
0 = \int_a^b f(x) \, dx = \sup_P L(P, f) \geq L(P^*, f) = \sum_{i=1}^n \left( \inf_{x_{i-1} \leq x \leq x_i} f(x) \right) \Delta x_i \geq \frac{f(y)}{2} \Delta x_k > 0,
\]
a contradiction.
Problem 4 (WR Ch 6 #4). If \( f(x) = 0 \) for all irrational \( x \), \( f(x) = 1 \) for all rational \( x \), prove that \( f \notin \mathbb{R} \) on \([a, b] \) for any \( a < b \).

Solution. For any partition \( P = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\} \), we have

\[
L(P, f) = \sum_{i=1}^{n} \left( \inf_{x_{i-1} \leq x \leq x_i} f(x) \right) \Delta x_i \quad \text{by the density of } \mathbb{Q}^c,
\]

\[
U(P, f) = \sum_{i=1}^{n} \left( \sup_{x_{i-1} \leq x \leq x_i} f(x) \right) \Delta x_i \quad \text{by the density of } \mathbb{Q}.
\]

\[
\int_a^b f = \sup_P L(P, f) = 0 \neq (b-a) = \inf_P U(P, f) = \int_a^b f,
\]

so \( f \notin \mathbb{R} \) on \([a, b] \).

Problem 5 (WR Ch 6 #5). Suppose \( f \) is a bounded real function on \([a, b] \), and \( f^2 \in \mathbb{R} \) on \([a, b] \). Does it follow that \( f \in \mathbb{R} \)? Does the answer change if we assume that \( f^3 \in \mathbb{R} \)?

Solution. In the first case, we have the following counterexample. Let

\[
f(x) = \begin{cases} 
1 & \text{if } x \in (\mathbb{Q}^c \cap [a, b]) \\
-1 & \text{if } x \in (\mathbb{Q} \cap [a, b]) 
\end{cases}
\]

Then by the previous proof with \(-1\) in place of \(0\), \( f \notin \mathbb{R} \) on \([a, b] \). But \( f^2 \equiv 1 \in \mathbb{R} \) on \([a, b] \). So it does not necessarily follow that if \( f^2 \in \mathbb{R} \) on \([a, b] \) then \( f \in \mathbb{R} \).

In the second case, it does necessarily follow that if \( f^3 \in \mathbb{R} \) on \([a, b] \) then \( f \in \mathbb{R} \) by the following proof. The reason this works for \( f^3 \) and not for \( f^2 \) is that the inverse of the cube function on \( \mathbb{R} \) is well-defined and is \( \phi(x) = \sqrt[3]{x} \). The square function does not have a well-defined inverse on all of \( \mathbb{R} \) (since if \( y = x^2 \) then \( x = \pm \sqrt{y} \)).

By Theorem 6.11, since \( \phi \) is continuous on all of \( \mathbb{R} \), then

\[
\phi(f^3(x)) = \sqrt[3]{f^3(x)} = f(x) \text{ is } \mathbb{R} \text{ on } [a, b].
\]

Problem 6 (WR Ch 6 #10). Let \( p \) and \( q \) be positive real numbers such that

\[
\frac{1}{p} + \frac{1}{q} = 1.
\]
Prove the following statements.

(a) If \( u \geq 0 \) and \( v \geq 0 \), then
\[
uv \leq \frac{u^p}{p} + \frac{v^q}{q}.
\]
Equality holds if and only if \( u^p = v^q \).

(b) If \( f, g \in \mathcal{R}(\alpha) \), \( f \geq 0 \), \( g \geq 0 \), and
\[
\int_a^b f^p \, d\alpha = \int_a^b g^q \, d\alpha = 1,
\]
then
\[
\int_a^b fg \, d\alpha = 1.
\]

(c) If \( f \) and \( g \) are complex functions in \( \mathcal{R}(\alpha) \), then
\[
\left| \int_a^b fg \, d\alpha \right| \leq \left\{ \int_a^b |f|^p \, d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q \, d\alpha \right\}^{1/q}.
\]
This is Hölder’s inequality. When \( p = q = 2 \) it is usually called the Schwarz inequality.

(d) Show that Hölder’s inequality is also true for the “improper” integrals described in Exercises 6.7 and 6.8.

**Solution.**

**Claim.** \( f(x) = e^x \) is a convex function.

Let \( x < t < y \). By the Mean Value Theorem, there exists some \( a \in (x, t) \) such that
\[
f(t) - f(x) = (t - x)f'(a)
\]
which means
\[
f'(a) = \frac{f(t) - f(x)}{t - x}.
\]

Once again, by the Mean Value Theorem, there exists some \( b \in (t, y) \) such that
\[
f(y) - f(t) = (y - t)f'(b)
\]
which means
\[
f'(b) = \frac{f(y) - f(t)}{y - t}.
\]

Notice that \( f''(x) = e^x > 0 \) for all \( x \in \mathbb{R} \). This means that \( f'(x) \) is strictly increasing. Therefore, since \( a < b \), we have \( f'(a) \leq f'(b) \), so
\[
\frac{f(t) - f(x)}{t - x} = f'(a) \leq f'(b) = \frac{f(y) - f(t)}{y - t}.
\]

Now for any \( \lambda \in (0, 1) \) we have \( x < (\lambda x + (1 - \lambda)y) < y \), so letting \( t = (\lambda x + (1 - \lambda)y) \) the above
inequality becomes
\[
\frac{f(t) - f(x)}{(\lambda x + (1 - \lambda)y) - x} \leq \frac{f(y) - f(t)}{y - (\lambda x + (1 - \lambda)y)}
\]
\[
\frac{f(t) - f(x)}{(1 - \lambda)\mu - \lambda x} \leq \frac{f(y) - f(t)}{\lambda(y - x)}
\]
\[
\lambda f(t) - \lambda f(x) \leq (1 - \lambda) f(y) - (1 - \lambda) f(t)
\]
\[
f(t) \leq \lambda f(x) + (1 - \lambda) f(y)
\]
\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y),
\]
so \(f(x) = e^x\) is convex.

(a) From here, we let \(\lambda = \frac{1}{p}\), so that \((1 - \lambda) = \frac{1}{q}\). The desired result is trivial if \(u = 0\) or \(v = 0\), so assume they are both strictly positive. Letting \(x = \log u^\theta\) and \(y = \log v^\theta\), the above inequality becomes
\[
e^{\frac{1}{p} \log u^\theta + \frac{1}{q} \log v^\theta} \leq \frac{1}{p} e^{\log u^\theta} + \frac{1}{q} e^{\log v^\theta}
\]
\[
e^{\log u + \log v} \leq \frac{u^\theta}{p} + \frac{v^\theta}{q}
\]
\[
u \leq \frac{u^\theta}{p} + \frac{v^\theta}{q}.
\]

(b) By part (a), for every \(x \in [a, b]\) we have
\[
f(x) g(x) \leq \frac{(f(x))^p}{p} + \frac{(g(x))^q}{q}.
\]
Therefore, taking integrals, we have
\[
\int_a^b f g \, d\alpha \leq \frac{\int_a^b f^p \, d\alpha}{p} + \frac{\int_a^b g^q \, d\alpha}{q} = \int_a^b f^p \, d\alpha + \int_a^b g^q \, d\alpha = 1.
\]

(c) If \(\int_a^b |f| \, d\alpha = 0\) or \(\int_a^b |g| \, d\alpha = 0\) the inequality is trivial. Otherwise, let \(A = \left\{\int_a^b |f|^p \, d\alpha\right\}^{1/p} > 0\) and let \(B = \left\{\int_a^b |g|^q \, d\alpha\right\}^{1/q} > 0\), and let
\[
F(x) = \frac{|f(x)|}{A} \quad \text{and} \quad G(x) = \frac{|g(x)|}{B}.
\]
These functions satisfy the hypotheses of part (b), so
\[
\int_a^b F G \, d\alpha \leq 1
\]
\[
\int_a^b |f| \frac{|g|}{A} \, d\alpha \leq 1
\]
\[
\int_a^b f g \, d\alpha \leq \int_a^b |f||g| \, d\alpha \leq AB = \left\{\int_a^b |f|^p \, d\alpha\right\}^{1/p} \left\{\int_a^b |g|^q \, d\alpha\right\}^{1/q}.
\]
(e) Since $x \to |x|$ and $x \to x^{1/p}$ and $x \to x^{1/q}$ are continuous functions (for $x > 0$), we have

$$\left| \int_0^1 fg \, da \right| = \left| \lim_{c \to 0} \int_c^1 fg \, da \right| = \left| \lim_{c \to 0} \int_c^1 fg \, da \right|$$

$$\leq \lim_{c \to 0} \left( \left\{ \int_c^1 |f|^p \, da \right\}^{1/p} \left\{ \int_c^1 |g|^q \, da \right\}^{1/q} \right)$$

$$= \left( \lim_{c \to 0} \left( \int_c^1 |f|^p \, da \right) \right)^{1/p} \left( \lim_{c \to 0} \left( \int_c^1 |g|^q \, da \right) \right)^{1/q}$$

$$= \left( \int_0^1 |f|^p \, da \right)^{1/p} \left( \int_0^1 |g|^q \, da \right)^{1/q}$$

assuming the integrals are all nonzero and finite. If they are not, the inequality is trivial. The proof follows similarly for $\int_a^\infty$. 
