Problem 1. Suppose $f$ is a real, continuous differentiable function on $[a, b]$ with $f(a) = f(b) = 0$, and $\int_a^b f^2(x) \, dx = 1$. Show that $\int_a^b x f(x) f'(x) \, dx = -1/2$.

Solution. By integration by parts with $u = x f(x)$, $dv = f'(x) \, dx$, we have $du = [x f'(x) + f(x)] \, dx$, $v = f(x)$, so

$$\int_a^b x f(x) f'(x) \, dx = \left[ x f^2(x) \right]_a^b - \int_a^b f(x) [x f'(x) + f(x)] \, dx$$

$$= -\int_a^b x f(x) f'(x) \, dx - \int_a^b f^2(x) \, dx$$

$$= -\int_a^b x f(x) f'(x) \, dx - \frac{1}{2}$$

$$2 \int_a^b x f(x) f'(x) \, dx = -1$$

$$\int_a^b x f(x) f'(x) \, dx = -1/2.$$
we define \( P_k \) by adding the points \((c_i - \frac{1}{n})\) and \((c_i + \frac{1}{n})\) for each \( 1 \leq i \leq m \) if they are in \([a, b]\). Let \( n_k \) be the number of points in \( P_k \). Then we have

\[
U(P_k, g) = \sum_{j=1}^{n_k-1} \left( \sup_{x_{j-1} \leq x \leq x_j} g(x) \right) \Delta x_j
\]

\[
\leq \sum_{j=1}^{n_k-1} \left( \sup_{x_{j-1} \leq x \leq x_j} f(x) \right) \Delta x_j + \sum_{3 \in \mathbb{C} \setminus \mathbb{L}} |g(c) - f(c)| \Delta x_j
\]

\[
\leq \sum_{j=1}^{n_k-1} \left( \sup_{x_{j-1} \leq x \leq x_j} f(x) \right) \Delta x_j + \frac{mM}{k}
\]

\[
= U(P_k, f) + \frac{mM}{k}.
\]

Choose the \( K \) to be the smallest integer larger than \( \frac{3mM}{c} \). That way \( \frac{mM}{k} < \frac{c}{3} \) for \( k \geq K \). Then for the lower Riemann sum we have:

\[
L(P_k, g) = \sum_{j=1}^{n_k-1} \left( \inf_{x_{j-1} \leq x \leq x_j} g(x) \right) \Delta x_j
\]

\[
\geq \sum_{j=1}^{n_k-1} \left( \inf_{x_{j-1} \leq x \leq x_j} f(x) \right) \Delta x_j - \sum_{3 \in \mathbb{C} \setminus \mathbb{L}} |g(c) - f(c)| \Delta x_j
\]

\[
\geq \sum_{j=1}^{n_k-1} \left( \inf_{x_{j-1} \leq x \leq x_j} f(x) \right) \Delta x_j - \frac{mM}{k}
\]

\[
= L(P_k, f) - \frac{mM}{k}.
\]

Because \( P_K \) is a refinement of \( P \), we have

\[
U(P_K, f) - L(P_K, f) \leq U(P, f) - L(P, f) < \epsilon,
\]

and finally putting this all together,

\[
U(P_K, g) - L(P_K, g) \leq \left( U(P_K, g) - U(P_K, f) \right) + \left( U(P_K, f) - L(P_K, f) \right) + \left( L(P_K, f) - L(P_K, g) \right)
\]

\[
< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]

Therefore \( g \in \mathcal{R}[a, b] \) by Theorem 6.6.

The result is not true if \( g(x) = f(x) \) for all \( x \) except for a countable number of points by the following counterexample. Let

\[
g(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{Q} \\
0 & \text{if } x \notin \mathbb{Q}
\end{cases}
\]

Then \( g \notin \mathcal{R}[a, b] \) but \( g(x) = 0 \) for except for a countable number of points (the rationals in \([a, b]\)) and \( 0 \notin \mathcal{R}[a, b] \).
Problem 3. Let $f : [0, \infty) \to \mathbb{R}$ be defined as $f(x) = 0$ if $0 \leq x \leq 1/2$ and $f(x) = 1$ if $1/2 \leq x < \infty$. Show that the function
\[ F(x) = \int_0^x f(t) \, dt, \]
defined for $0 \leq x < \infty$, is differentiable for $x \neq 1/2$ and is not differentiable for $x = 1/2$.

Solution. $f$ is continuous except at $x = 1/2$. By Theorem 6.20, this means that $F$ is differentiable everywhere except possibly at $x = 1/2$, and $F'(x) = f(x)$ when $x \neq 1/2$.

\[
\lim_{h \to 0^+} \frac{F(1/2 + h) - F(1/2)}{h} = \lim_{h \to 0^+} h \int_{1/2}^{1/2+h} f(t) \, dt \quad = \lim_{h \to 0^+} \frac{1}{h} \int_{1/2}^{1/2+h} f(t) \, dt \quad = \lim_{h \to 0^+} \frac{1}{h} 1 \quad = 1.
\]

\[
\lim_{h \to 0^-} \frac{F(1/2 + h) - F(1/2)}{h} = \lim_{h \to 0^-} h \int_{1/2}^{1/2+h} f(t) \, dt \quad \quad = \lim_{h \to 0^-} \frac{1}{h} \int_{1/2}^{1/2+h} f(t) \, dt \quad \quad = \lim_{h \to 0^-} \frac{1}{h} 0 \quad = 0.
\]

Therefore the limit does not exist, so $F$ is not differentiable at $x = 1/2$.

Problem 4. Prove that if $f$ and $g$ are Riemann integrable on $[a, b]$ (i.e. $f, g \in \mathcal{R}[a, b]$) and there exists $N > 0$ such that $g(x) \geq 1/N$ for all $x \in [a, b]$, then $f/g \in \mathcal{R}[a, b]$.

Solution. Let $g$ be bounded above by $M$ on $[a, b]$. By Theorem 6.11, since $\phi(x) = \frac{1}{x}$ is continuous on $[\frac{1}{M}, M]$ (because it doesn’t contain 0), then $\phi(g(x)) = \frac{1}{g(x)}$ is Riemann integrable on $[a, b]$. Now by Theorem 6.13(a), $f/g = f \cdot \frac{1}{g}$ is Riemann integrable on $[a, b]$.}

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