**Problem 1** (WR Ch 3 #21). Prove the following analogue of Theorem 3.10(b): If \( \{E_n\} \) is a sequence of closed nonempty and bounded sets in a complete metric space \( X \), if \( E_n \supset E_{n+1} \), and if \( \lim_{n \to \infty} \text{diam}(E_n) = 0 \), then \( \bigcap_{n=1}^{\infty} E_n \) consists of exactly one point.

**Solution.** If \( \bigcap E_n \) has two or more points, say \( x, y \in \bigcap E_n \) with \( x \neq y \), then \( x, y \in E_n \) for all \( n \), so \( \text{diam}(E_n) \geq d(x, y) > 0 \), contradicting the fact that \( \text{diam}(E_n) \to 0 \). Therefore, we know it has at most one point; all that is left is to prove it is nonempty.

For each \( n \), pick a point \( x_n \) in \( E_n \) (which is nonempty). Since \( E_n \supset E_{n+1} \), we know \( \{x_n, x_{n+1}, \ldots\} \subset E_n \). The fact that \( \text{diam}(E_n) \to 0 \) implies that for any \( \epsilon > 0 \) we can pick some \( N \in \mathbb{N} \) such that \( \text{diam}(E_n) < \epsilon \) for \( n \geq N \), which means for any \( n, m \geq N \) we have

\[
|x_n - x_m| \leq \text{diam}(\{x_N, x_{N+1}, x_{N+2}, \ldots\}) \leq \text{diam}(E_N) < \epsilon,
\]

so \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is complete, that means that \( \{x_n\} \) converges, say \( x_n \to x \). \( E_n \) is closed and it contains \( \{x_n, x_{n+1}, \ldots\} \), (i.e. the tail of the sequence \( \{x_n\} \)) so \( x \in E_n \) for all \( n \). Therefore, \( x \in \bigcap E_n \), so it is nonempty, finishing the proof.

**Problem 2** (WR Ch 3 #22). Suppose \( X \) is a nonempty complete metric space, and \( \{G_n\} \) is a sequence of dense open subsets of \( X \). Prove Baire's theorem, namely, that \( \bigcap_{n=1}^{\infty} G_n \) is not empty. (In fact, it is dense in \( X \).)

**Solution.** Let \( A = \bigcap_{n=1}^{\infty} G_n \). We will prove \( A \) is dense, which will show it is nonempty (since if \( A = \emptyset \) and \( A \) is dense then \( X = \overline{A} = \emptyset \), contradicting the fact that \( X \) is nonempty). To do that, we'll first prove the following claim:

**Claim.** \( A \subset X \) is dense iff \( N \cap A \neq \emptyset \) for every neighborhood \( N \subset X \).

**Proof of claim.**

\( \implies \) Assuming \( A \subset X \) is dense, then \( A \cup A' = \overline{A} = X \). Then for any \( x \in N \), there are two possible cases:

(a) \( x \in A \), in which case \( x \in N \cap A \), so \( N \cap A \neq \emptyset \).

(b) \( x \notin A \) and \( x \in A' \). Then since \( x \) is a limit point of \( A \), and \( N \) is a neighborhood of \( x \), there is some \( y \in N \) such that \( y \neq x \) and \( y \in A \). But then \( y \in N \cap A \), so \( N \cap A \neq \emptyset \).
Assuming \( N \cap A \neq \emptyset \) for every neighborhood \( N \subset X \), we want to show that \( \overline{A} = X \). Let \( x \in X \).

If \( x \in A \) we are done. Otherwise, assume \( x \notin A \), and we must show that \( x \) is a limit point of \( A \). For any neighborhood \( N \) of \( x \), we know that \( N \cap A \neq \emptyset \), so there must be some \( y \in N \cap A \).

However, since \( x \notin A \), that means that \( y \neq x \). Therefore \( x \) is a limit point of \( A \), finishing the proof of the claim.

Now, using the claim, let \( N_0 \) be any neighborhood in \( X \), and we’ll show that \( N_0 \cap A \neq \emptyset \). Since \( G_1 \) is a dense open subset of \( X \), we know that \( N \cap G_1 \neq \emptyset \), so take any \( x_1 \in N_0 \cap G_1 \). Since \( N_0 \cap G_1 \) is open, there is a neighborhood \( N_1 \) of \( x_1 \), and we can shrink \( N_1 \) if necessary to guarantee that \( \text{diam} \, N_1 \leq 1 \). Let \( E_1 = N_1 \). We repeat this process, so that if \( x_{n-1} , N_{n-1} \) and \( E_{n-1} \) are already defined, we define the next terms in the sequence as follows: since \( G_n \) is a dense open subset of \( X \), we know that \( N_{n-1} \cap G_n \neq \emptyset \), so take any \( x_n \in N_{n-1} \cap G_n \). Since \( N_{n-1} \cap G_n \) is open, there is a neighborhood \( N_n \) of \( x_n \), and we can shrink \( N_n \) if necessary to guarantee that \( \text{diam} \, N_n \leq \frac{1}{n} \). Let \( E_n = N_n \). Then \( \{E_n\} \) is a collection satisfying the conditions of the previous exercise, so \( \bigcap E_n = \{p\} \) for some point \( p \in N_0 \), and since \( E_n \subset G_n \), that means there is a point \( p \) in each \( G_n \), and thus \( p \in A = \bigcap G_n \). Therefore, \( N_0 \cap A \neq \emptyset \).

**Problem 3** (WR Ch 3 #23). Suppose \( \{p_n\} \) and \( \{q_n\} \) are Cauchy sequences in a metric space \( X \). Show that the sequence \( \{d(p_n, q_n)\} \) converges.

**Solution.** For any \( m, n \),

\[
d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_n) + d(q_n, q_m) \implies d(p_n, q_n) - d(p_m, q_m) \leq d(p_n, p_m) + d(q_n, q_m).
\]

Doing the same argument but switching \( m \) and \( n \), we also have

\[
d(p_m, q_m) - d(p_n, q_n) \leq d(p_m, p_n) + d(q_n, q_m),
\]

so putting these two together we have

\[
|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_n, q_m).
\]

For any \( \epsilon > 0 \), since \( \{p_n\} \) is Cauchy, we can find an \( N_1 \in \mathbb{N} \) such that for \( m, n \geq N_1 \), \( d(p_n, p_m) < \epsilon/2 \), and since \( \{q_n\} \) is Cauchy, we can find an \( N_2 \in \mathbb{N} \) such that for \( m, n \geq N_2 \), \( d(q_n, q_m) < \epsilon/2 \). Therefore, for \( n, m \geq N = \max(N_1, N_2) \) we have

\[
|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_n, q_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\]

so \( \{d(p_n, q_n)\} \) is a Cauchy sequence, and since \( \mathbb{R} \) is a complete metric space and \( \{d(p_n, q_n)\} \) is a Cauchy sequence of real numbers, it converges.

**Problem 4** (WR Ch 4 #1). Suppose \( f \) is a real function defined on \( \mathbb{R}^1 \) which satisfies

\[
\lim_{h \to 0} |f(x + h) - f(x - h)| = 0
\]

for every \( x \in \mathbb{R}^1 \). Does this imply that \( f \) is continuous?
Solution. It does not imply that \( f \) is continuous. The following is a counterexample.

\[
 f(x) = \begin{cases} 
 1 & \text{if } x=0 \\
 0 & \text{otherwise}
\end{cases}
\]

Notice that for \( \epsilon = \frac{1}{2} \), for any \( \delta > 0 \) and any \( x \) such that \( 0 < |0 - x| < \delta \) (i.e. \( x \neq 0 \)), we have

\[
| f(x) - f(0) | = |0 - 1| = 1 > \frac{1}{2},
\]

so \( f \) is discontinuous at 0.

However, for any \( x \neq 0 \), note that for any \( h < |x| \), \( x + h \neq 0 \) and \( x - h \neq 0 \), so

\[
 f(x + h) - f(x - h) = 0 - 0 = 0,
\]

and thus \( \lim_{h \to 0} | f(x + h) - f(x - h) | = 0 \). If \( x = 0 \), then for any \( h > 0 \),

\[
 f(x + h) - f(x - h) = 0 - 0 = 0,
\]

and thus \( \lim_{h \to 0} | f(x + h) - f(x - h) | = 0 \) again. This finishes the proof of the counterexample.

**Problem 5** (WR Ch 4 #7). If \( E \subset X \) and \( f \) is a function defined on \( X \), the restriction of \( f \) to \( E \) is the function \( g \) whose domain of definition is \( E \), such that \( g(p) = f(p) \) for \( p \in E \). Define \( f \) and \( g \) on \( \mathbb{R}^2 \) by: \( f(0,0) = g(0,0) = 0 \), \( f(x,y) = xy^2/(x^2 + y^4) \), \( g(x,y) = xy^2/(x^2 + y^6) \) if \( (x,y) \neq (0,0) \). Prove that \( f \) is bounded on \( \mathbb{R}^2 \), that \( g \) is unbounded in every neighborhood of \( (0,0) \), and that \( f \) is not continuous at \( (0,0) \); nevertheless, the restrictions of both \( f \) and \( g \) to every straight line in \( \mathbb{R}^2 \) are continuous!

**Solution.** \( f \) is bounded on \( \mathbb{R}^2 \) because

\[
(|x| - y^2)^2 \geq 0 \iff x^2 - 2|x|y^2 + y^4 \geq 0 \iff x^2 + y^4 \geq 2|x|y^2 \iff \frac{1}{2} \geq \left| \frac{xy^2}{x^2 + y^4} \right|.
\]

\( g \) is unbounded in every neighborhood of \( (0,0) \) because if we let \( x_n = \frac{1}{n^2} \) and \( y_n = \frac{1}{n} \) then \( (x_n, y_n) \to (0,0) \) and

\[
g(x_n, y_n) = \frac{1}{n^4} \frac{1}{n^2} \frac{1}{n^2} = \frac{1}{n^4} \frac{1}{n^2} = \frac{n^6}{2} \to \infty.
\]

This is because \( (x_n, y_n) \to (0,0) \) implies that for any neighborhood \( U \) of \( (0,0) \), there is some \( N \in \mathbb{N} \) such that \( (x_n, y_n) \in U \) for \( n \geq N \), and \( g(x_n, y_n) \) gets arbitrarily large. Next, we show \( f \) is not continuous at \( p = (0,0) \) simply by using Theorems 4.2 and Theorem 4.6 to say

\[
f \text{ is continuous at } p \iff 4.6 \lim_{x \to p} f(x) = f(p) \iff 4.2 f(p_n) \to f(p) \text{ for every sequence } p_n \to p.
\]

So to show \( f \) is not continuous, we only need to find one sequence \( p_n \to p \) such that \( f(p_n) \neq f(p) \).

Let \( x_n = \frac{1}{n^2} \) and \( y_n = \frac{1}{n} \) and put \( p_n = (x_n, y_n) \), so that \( p_n \to (0,0) \). But then

\[
f(p_n) = f(x_n, y_n) = \frac{1}{n^4} \frac{1}{n^2} = \frac{1}{n^2} \frac{1}{n^2} = \frac{1}{2} \neq 0 = f(0,0) = f(p),
\]
Lastly, we show that the restriction of \( f \) and \( g \) to straight lines in \( \mathbb{R}^2 \) are continuous functions. First of all, \( f \) and \( g \) are continuous on \( \mathbb{R}^2 \setminus \{(0,0)\} \) because they are composed of addition, multiplication, and division with a denominator that is nonzero on \( \mathbb{R}^2 \setminus \{(0,0)\} \). Therefore \( f \) and \( g \) restricted to any line that does not go through \((0,0)\) are already continuous. What we need to check is that they are continuous at \((0,0)\) for any line going through \((0,0)\). There are two lines of this form: \( y = mx \) and \( x = 0 \). Along the line \( x = 0 \), \( f \) and \( g \) are constantly 0, so they are continuous. Along the line \( y = mx \), for \( x \neq 0 \), we have

\[
f(x, mx) = \frac{x(mx)^2}{x^2 + (mx)^4} = \frac{m^2x}{1 + m^4x^2} \to 0 = f(0,0) \quad \text{as } x \to 0,
\]

and

\[
g(x, mx) = \frac{x(mx)^2}{x^2 + (mx)^6} = \frac{m^2x}{1 + m^6x^2} \to 0 = g(0,0) \quad \text{as } x \to 0,
\]

so \( f \) and \( g \) are continuous at \((0,0)\) when restricted to straight lines.

**Problem 6** (WR Ch 4 #8). Let \( f \) be a real uniformly continuous function on the bounded set \( E \) in \( \mathbb{R}^1 \). Prove that \( f \) is bounded on \( E \).

Show that the conclusion is false if boundedness of \( E \) is omitted from the hypothesis.

**Solution.** Since \( E \) is a bounded set in \( \mathbb{R} \), there is some \( R > 0 \) such that \( E \subset [-R,R] \). Set \( \epsilon = 1 \). Since \( f \) is uniformly continuous, there is some \( \delta > 0 \) such that \( |x - y| < \delta \) implies \( |f(x) - f(y)| < \epsilon = 1 \). Let \( n \in \mathbb{N} \) be the smallest integer such that \( \frac{1}{n} < \delta \), and let

\[
[-R, -R + \frac{1}{n}], [-R + \frac{1}{n}, -R + \frac{2}{n}], \ldots, [R - \frac{2}{n}, R - \frac{1}{n}], [R - \frac{1}{n}, R]
\]

be a collection \( S \) of \( 2nR \) intervals that exactly line up and cover \([-R, R]\). By omitting some if necessary, let \( I_1, I_2, \ldots, I_N \) be those intervals from the collection that overlap with \( E \), i.e. such that \( I_k \cap E \neq \emptyset \) for \( 1 \leq k \leq N \). Now for each \( k \) from 1 to \( N \), pick some point \( x_k \in I_k \cap E \). Then we have

\[
x_k, x \in I_k \cap E \implies |x_k - x| < \frac{1}{n} < \delta \implies |f(x_k) - f(x)| < 1 \implies |f(x)| < 1 + |f(x_k)|.
\]

Letting \( M = \max_{1 \leq k \leq N} (1 + f(x_k)) \), for any \( x \in E \) we have \( |f(x)| < M \), so \( f \) is a bounded function on \( E \).

The second part asks us to show the conclusion is false if boundedness of \( E \) is omitted from the hypothesis. Let \( E = \mathbb{R} \), which is unbounded, and let \( f(x) \), which is uniformly continuous because for any \( \epsilon > 0 \), let \( \delta = \epsilon \) and we have

\[
|x-y| < \delta \implies |f(x) - f(y)| = |x-y| < \delta = \epsilon.
\]

However, \( f(x) = x \) is not bounded on \( E = \mathbb{R} \).