Problem set 4

Do for Friday, April 27

1. Suppose \( f \) is a real, continuously differentiable function on \([a,b]\) with \( f(a)=f(b)=0 \), and
\[
\int_a^b f^2(x) \, dx = 1.
\]
Show that
\[
\int_a^b xf(x)f'(x) \, dx = -\frac{1}{2}.
\]

2. Prove that if \( f \in \mathcal{R}[a,b] \) and \( g \) is a function for which \( g(x) = f(x) \) for all \( x \) except for a finite number of points, then \( g \) is Riemann integrable. Is the result still true if \( g(x) = f(x) \) for all \( x \) except for a countable number of points?

3. Let \( f : [0, \infty) \to \mathbb{R} \) be defined as \( f(x) = 0 \) if \( 0 \leq x \leq 1/2 \) and \( f(x) = 1 \) if \( 1/2 < x \leq \infty \).
Show that the function
\[
F(x) = \int_0^x f(t) \, dt,
\]
defined for \( 0 \leq x < \infty \), is differentiable for \( x \neq 1/2 \) and is not differentiable for \( x = 1/2 \).

4. Prove that if \( f \) and \( g \) are Riemann integrable on \([a,b]\) (i.e. \( f, g \in \mathcal{R}[a,b] \)) and there exists \( N > 0 \) such that \( g(x) \geq 1/N \) for all \( x \in [a,b] \), then \( f/g \in \mathcal{R}[a,b] \). (Hint: First show \( 1/g \in \mathcal{R}[a,b] \), and then use Theorem 6.13.)