Problem set 6

Do for Friday, May 11.

1. Prove that the family \( \{ \sin nx, n = 1, 2, 3 \ldots \} \) is not equicontinuous on the interval \([-1, 1]\).

2. Prove that the family of all polynomials of degree \( \leq N \) with coefficients in the interval \([-1, 1]\) is uniformly bounded and equicontinuous on any compact interval.

3. For any continuous, real valued function \( f \) on \([0, 1]\), let \( F_f(x) = \int_0^x f(t) \, dt \). Show that the set of functions
   \[ \mathcal{F} = \{ F_f : \|f\| \leq 1 \} \]
   is bounded and equicontinuous.

4. Give an example of a metric space \( X \) and a sequence of functions \( \{f_n\} \) on \( X \) such that \( \{f_n\} \) is equicontinuous but not uniformly bounded.

5. Give an example of a uniformly bounded and equicontinuous sequence of functions on \( \mathbb{R} \) which does not have any uniformly convergent subsequences.

6. Let \( X \) be a metric space such that \( X = \bigcup_{n=1}^{\infty} K_n \), where each \( K_n \) is compact and such that any bounded open set \( U \) is contained in \( K_n \) for some \( n \). (An example is \( X := \mathbb{R}^k \) with \( K_n := \{ x \in \mathbb{R}^n : \|x\| \leq n \} \).)
   Let \( \{f_j\} \) be a sequence of functions which are pointwise bounded on \( X \) and whose restriction to any \( K_n \) is equicontinuous. Show that there exists a subsequence \( \{f_{n_j}\} \) that converges to a continuous function on \( X \).
   Hint: Use a diagonal trick.