#11: Show that \( \{\text{cis} \, k : k \text{ a non-negative integer} \} \) is dense in \( T = \{z \in \mathbb{C} : |z| = 1\} \). For which values of \( \theta \) is \( \{\text{cis}(k\theta) : k \text{ a non-negative integer} \} \) dense in \( T \)?

To show that \( \{\text{cis} \, k : k \text{ a non-negative integer} \} \) is dense in \( T \), we need only show that the closure of \( \{\text{cis} \, k : k \text{ a non-negative integer} \} = T \). Observe that cis \( k \) consists of infinitely many points on the unit circle. (Repetition could only occur if \( k = j + 2\pi l \), for some integer \( l \) and \( 2\pi \notin \mathbb{Z}. \)) We claim that the closure of that set will consist of the unit circle itself, which is precisely \( T \).

To prove this claim, though, we shall prove a stronger statement. Namely, we shall prove:

The set \( \{\text{cis}(2\pi k \theta)\} \) is dense in \( T \) if and only if \( \theta \) is irrational. \((*)\)

Notice that in the case where \( \theta = 1/2\pi \), we have precisely the problem above.

Define \( \text{frac}(x) = x - \lfloor x \rfloor \), where \( \lfloor x \rfloor \) denotes the floor of \( x \).

Showing \((*)\), is equivalent to showing the following:

The sequence \( \text{frac}(\theta), \text{frac}(2\theta), \ldots \) is dense in \([0, 1)\) if and only if \( \theta \) is irrational. \((**)\)

The equivalence follows from the fact that cis \(2\pi k \theta\) is determined by the value \(2\pi k \theta\). So, we essentially want to show that the angle is dense on the interval \([0, 2\pi)\). However, we need to consider \(2\pi k \theta \pmod{2\pi}\), since the unit circle wraps around.

Dividing through by \(2\pi\), we are left with showing that \( k \theta \pmod{1} \) is dense in \([0, 1)\), which is precisely what \((**)\) looks to be establish with the same if and only if statement.

Before we prove \((**\), we shall prove some lemmas.

**Lemma 1:** If \( \theta \) is irrational and \( n \neq m \), then \( \text{frac}(n\theta) \neq \text{frac}(m\theta) \).

**Proof:** Suppose not. That is, suppose \( \text{frac}(n\theta) = \text{frac}(m\theta) \). Then we have that \( n\theta - \lfloor n\theta \rfloor = m\theta - \lfloor m\theta \rfloor \), i.e. \((n - m)\theta = \lfloor n\theta \rfloor - \lfloor m\theta \rfloor \), i.e. \( \theta = \frac{\lfloor n\theta \rfloor - \lfloor m\theta \rfloor}{n - m} \in \mathbb{Q} \), which is a contradiction to \( \theta \) being irrational, which completes the proof. Note: We used the fact that \( n \neq m \) to ensure that the denominator is non-zero. \(\blacksquare\)
Lemma 2: Suppose $\theta \in [0, 1)$ is irrational. Then for all $\epsilon > 0$, there exists $h, k \in \mathbb{Z}^+$ such that $|k\theta - h| < \epsilon$.

Proof: Subdivide $[0, 1)$ into half-open intervals of equal length less than $\epsilon$. By Lemma 1, two of the fractional parts lie in the same subinterval. That is, there exists $n, m \in \mathbb{Z}^+$ (with $n \neq m$) such that $|\text{frac}(n\theta) - \text{frac}(m\theta)| < \epsilon$. (This follows from the Pigeonhole Principle.)

Thus, we have that $|(n-m)\theta - (\lfloor n\theta \rfloor - \lfloor m\theta \rfloor)| < \epsilon$. Letting $k = n - m$ and $h = \lfloor n\theta \rfloor - \lfloor m\theta \rfloor$, we complete the proof. ■

Now we have all of the necessary machinery to prove (**).

One direction is trivial. If $\theta \in \mathbb{Q}$, then $\{\text{frac}(n\theta): n \in \mathbb{Z}^+\}$ is finite and thus cannot be dense in $T$. (A finite subset of $\mathbb{C}$ is closed and therefore cannot be dense in an infinite set.)

Suppose $\theta$ is irrational. Without loss of generality, we may suppose $\theta \in [0, 1)$. Indeed, we may replace $\theta$ with $\text{frac}(\theta)$ after noticing that $\text{frac}(n\theta) = \text{frac}(n \text{frac}(\theta))$.

Let $\epsilon > 0$ be given and $\alpha \in [0, 1)$. We need to find an integer $m$ such that $|\text{frac}(m\theta) - \alpha| < \epsilon$, since this will show that $\{\text{frac}(n\theta): n \in \mathbb{Z}^+\}$ is dense in $[0, 1)$.

Using the notation from Lemma 2, suppose that $k\theta > h$. (The argument is almost identical in the case $k\theta < h$.)

Choose the largest $N \in \mathbb{Z}^+$ such that $\text{frac}(k\theta) < 1/N$. Consider the sequence:

$0, \text{frac}(k\theta), \text{frac}(2k\theta), \ldots, \text{frac}(Nk\theta)$.

In light of $\text{frac}(mk\theta) = m\text{frac}(k\theta)$ if and only if $\text{frac}(k\theta) < 1/m$, we see that the above sequence is increasing and equally-spaced with common difference $\text{frac}(k\theta)$.

(The equal spacing is readily apparent and the increasing property follows from the fact that the sequence is never greater than 1.)

Thus, we have subdivided the unit interval into $N + 1$ subintervals with the following partition: $[0, \text{frac}(k\theta)), [\text{frac}(k\theta), \text{frac}(2k\theta)), \ldots, [\text{frac}(Nk\theta), 1)$.

Since $\alpha$ lies in one of the subintervals above, it suffices to show the following claim.
Claim: \( [0, \frac{k}{q}), \frac{2k}{q}, \ldots \frac{Nk}{q}, 1) \) divides \([0, 1)\) into subintervals of length less than \( \epsilon \).

**Proof:** By the proof of Lemma 2, we have that the first \( N \) subintervals have length \( \frac{k}{q} < \epsilon \). Thus, we need only consider the last subinterval.

By the def. of \( N \), we have that
\[
N \frac{k}{q} > \frac{N}{N+1} = 1 - \frac{1}{N+1} .
\]
This implies that
\[
0 < 1 - \frac{Nk}{q} < \frac{1}{N+1} < \frac{k}{q} < \epsilon ,
\]
proving the claim and completing the proof. ■

Thus, we have shown that the sequence \( \frac{q}{k}, \frac{2q}{k}, \ldots \) is dense in \([0, 1)\) if and only if \( \theta \) is irrational. From our discussion earlier, though, this is equivalent to showing that the set \( \{\text{cis}(2\pi k\theta)\} \) is dense in \( T \) if and only if \( \theta \) is irrational. ■

By the proof of the slightly more general theorem, we see that \( \{\text{cis}(k\theta)\} \) will be dense in \( T \) provided \( \theta \neq 0 \) or \( \theta \neq m\pi/n \), where \( m, n \in \mathbb{Z} \). This ensures that then \( k\theta \) will never be equal to \( 2\pi \), thereby ensuring that \( \text{cis}(k\theta) \) maps to infinitely many points on the unit circle. And by the proof of (**), we have that the set will be dense in \( T \). ■

**Pg 17**

**#4:** Prove the following generalization of Lemma 2.6. If \( \{D_j: j \in J\} \) is a collection of connected subsets of \( X \) and if for each \( j \) and \( k \) in \( J \) we have \( D_j \cap D_k \neq \emptyset \), then \( D = \bigcup \{D_j: j \in J\} \) is connected.

If we suppose that \( J \) is countable (which may not be the case), then we can apply the following inductive argument. A general proof for the case when \( J \) may not be countable appears afterwards.

For this first (incomplete) proof, suppose that \( J \) is countable. We shall proceed via induction on \( |J| \), the number of \( D_j \). Let \( S = \{\{D_j: j \in J\} \) is a collection of connected subsets of \( X \) and if for each \( j \) and \( k \) in \( J \) we have \( D_j \cap D_k \neq \emptyset \), then \( D = \bigcup \{D_j: j \in J\} \) is connected\}.\]

Notice that \( 1 \in S \), trivially. We also have that \( 2 \in S \), since if \( D_1 \cap D_2 \neq \emptyset \), then there exists a point \( x_{1,2} \in D_1 \cap D_2 \). By Lemma 2.6, we have that \( D = D_1 \cup D_2 \) is connected.
Suppose that \( n \in S \) for some \( n \). We want to show that \( n + 1 \in S \) as well. Consider a collection \( \{D_j : j \in J\} \) of connected subsets of \( \mathcal{X} \), where \( |J| = n + 1 \). Consider \( D = \bigcup \{D_j : j \in J\} \). Let \( D_{1\ldots n} = D_1 \cup D_2 \cup \ldots \cup D_n \). By the inductive hypothesis, we have that \( D_{1\ldots n} \) is connected. We can rewrite \( D \) as \( D = D_{1\ldots n} \cup D_{n+1} \).

Since \( D_j \cap D_k \neq \emptyset \) for all \( j, k \), we have that \( D_{1\ldots n} \cap D_{n+1} \neq \emptyset \). Thus, there exists a point \( x_{1\ldots n,n+1} \in D_{1\ldots n} \cap D_{n+1} \). Again, by Lemma 2.6, we have that \( D_{1\ldots n} \cup D_{n+1} \) is connected, which completes the inductive hypothesis and hence the proof.

Essentially, we can draw a series of line segments through each \( x_{jk} \), passing through the \( D_j \), thus connecting them. ■

![Figure 1](image-url): An illustration of how the \( D_j \) may be connected via line segments

Now, suppose we do not have any restriction on \( J \). That is, \( J \) is only an index set.

Let \( A \) be a subset of the metric space \((\mathcal{D}, d)\) that is both open and closed and suppose that \( A \neq \emptyset \). Our goal is to show that \( A = \mathcal{D} \), since that implies that \( \mathcal{D} \) is connected.

\( A \cap D_j \) is open in \((\mathcal{D}_j, d)\) for each \( j \) and also \( A \cap D_j \) is closed in \((\mathcal{D}_j, d)\). (These statements follow from Exercises 1.8 and 1.9, respectively.)

Since \( D_j \) is connected, then either \( A \cap D_j = \emptyset \) or \( A \cap D_j = D_j \). Since \( A \neq \emptyset \), we know that there exists at least one \( k \) such that \( A \cap D_k \neq \emptyset \), i.e. \( A \cap D_k = D_k \).

By hypothesis, there exists \( x_{jk} \in D_j \cap D_k \) for all \( j, k \in J \).

This implies that \( x_{jk} \in D_k = A \cap D_k \) and \( x_{jk} \in D_j \), i.e. \( x_{jk} \in A \cap D_k \cap D_j \), which implies that \( x_{jk} \in A \cap D_j \).

Hence, we have that \( A \cap D_j \neq \emptyset \). Thus, it must be that \( A \cap D_j = D_j \). We can continue this process for each index \( j \). Thus, \( A \cap D_j = D_j \), i.e. \( D_j \subseteq A \) for each index \( j \).

This gives that \( D = A \), which shows that \( D \) is connected. ■
#5: Show that if $F \subset X$ is closed and connected, then for every pair of points $a, b$ in $F$ and each $\epsilon > 0$, there are points $z_0, z_1, \ldots, z_n$ in $F$ with $z_0 = a, z_n = b$ and $d(z_{k-1}, z_k) < \epsilon$ for $1 \leq k \leq n$. Is the hypothesis that $F$ be closed needed? If $F$ is a set which satisfies this property then $F$ is not necessarily connected, even if $F$ is closed. Give an example to illustrate this.

Fix $a$. Let $\epsilon > 0$. Consider $A = \left\{ z \in F : \exists \{z_j\}_{j=0}^n \text{ with } z_0 = a, \ldots, z_n = z \text{ s.t. } d(z_j, z_{j+1}) < \epsilon' \right\}$. Our goal is to show that $A = F$, so we need to show $A$ is both open and closed in $F$ and $A \neq \emptyset$. Observe that $a \in A$, so we have that $A \neq \emptyset$.

To show $A$ is open in $F$, consider $B(z, \epsilon)$. We want to show that $B(z, \epsilon) \cap F \subseteq A$. This follows from the fact that if we can find a path from $a$ to $z$, then we can find a path to a point, say $x$, in $B(z, \epsilon)$ by adding one more path from $z$ to $x$. Since $d(z, x) < \epsilon$, we have that the conditions of inclusion in $A$ are met. Thus, $A$ is open in $F$.

To show that $A$ is closed in $F$, we shall show that $A^C \subseteq F$ is open. Given $x \in A^C$, we need to show that $B(x, \epsilon) \subseteq A^C$. Consider $y \in B(x, \epsilon)$. We claim that $y \in A^C$.

Suppose not. Then $y \in A$. Since $d(x, y) < \epsilon$, by the same argument for why $A$ is open, we have that there is a path from $a$ to $x$ via $y$, hence $x \in A$. But this is a contradiction, hence $y \in A^C$, which implies that $B(x, \epsilon) \subseteq A^C$, i.e. $A^C$ is open in $F$. Thus $A$ is also closed in $F$. So, we conclude that $A = F$ and hence connected.

Note: We did not need the hypothesis that $F$ is closed in this proof.

As a counterexample, consider $F = \{0\} \cup \left\{ \frac{2k+1}{2^j} : j, k \in \mathbb{Z^+}, \ 2k+1 \leq 2^j \right\}$. For any $\epsilon > 0$, we can find dyadic rationals such that $d(z_i, z_{i+1}) < \epsilon$, but $F$ is not connected. ■

Pg 20

#5: Show that every convergent sequence in $(X, d)$ is a Cauchy sequence.

Suppose $\{x_n\} \xrightarrow{\text{w}} x$ is a convergent sequence in $(X, d)$. Then for every $\epsilon > 0$, there exists an integer $N$ such that $d(x, x_n) < \epsilon/2$ whenever $n \geq N$.

Recall, a sequence $\{x_n\}$ is a Cauchy sequence if for every $\epsilon > 0$, there is an integer $N$ such that $d(x_n, x_m) < \epsilon$, for all $n, m \geq N$.

Consider $d(x_n, x_m)$. By the fourth property of metric spaces (the triangle inequality), we have $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \epsilon/2 + \epsilon/2 = \epsilon$, for all $n, m \geq N$, as desired. ■
#8: Suppose \( \{x_n\} \) is a Cauchy sequence and \( \{x_{n_k}\} \) is a subsequence that is convergent. Show that \( \{x_n\} \) must be convergent.

Suppose \( \{x_n\} \) is a Cauchy sequence and \( \{x_{n_k}\} \) is a subsequence that is convergent, say to \( x \).

By the definition of convergence, for all \( \epsilon > 0 \), there exists an integer \( K \) such that for all \( n_k \geq K \), \( d(x_{n_k}, x) < \epsilon/2 \).

By the definition of a Cauchy sequence, there exists an integer \( N \) such that for all \( n, n_k \geq N \), \( d(x_n, x_{n_k}) < \epsilon/2 \).

Let \( N' = \max\{K, N\} \)

For all \( n \geq N' \), we have \( d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \epsilon/2 + \epsilon/2 = \epsilon \), where the first \( \epsilon/2 \) comes from the fact that we have a Cauchy sequence and the second \( \epsilon/2 \) comes from the fact that we have a convergent sequence.

Thus, we have that for all \( \epsilon > 0 \), there exists an integer \( N' \) such that \( d(x_n, x) < \epsilon \) whenever \( n \geq N' \).
Additional Problems to Look At and Know

Pg 13

#2: Which of the following subsets of \( \mathbb{C} \) are open and which are closed: (a) \( \{ z : |z| < 1 \} \); (b) the real axis; (c) \( \{ z : z^n = 1 \text{ for some integer } n \geq 1 \} \); (d) \( \{ z \in \mathbb{C} : z \text{ is real and } 0 \leq z < 1 \} \); (e) \( \{ z \in \mathbb{C} : z \text{ is real and } 0 \leq z \leq 1 \} \)?

(a) \( \{ z : |z| < 1 \} \) is open. ■

(b) the real axis is closed; \( x \in (-\infty, \infty) \). (It is not open since any ball is not completely contained in the interval.) It is closed because a limit of real numbers converges to a real number. ■

(c) \( \{ z : z^n = 1 \text{ for some integer } n \geq 1 \} \) is closed, since it consists of a finite set of points. If \( z = r \text{cis} \theta \), then \( z^n = r^n \text{cis}(n\theta) \). Since \( z^n = 1 \), we have that \( z^n = \text{cis}(n\theta) \), which consists only of (finitely many, actually \( n \)) points on the unit circle. ■

(d) \( \{ z \in \mathbb{C} : z \text{ is real and } 0 \leq z < 1 \} \) is neither open or closed, since the left endpoint is included but the right endpoint is not. ■

(e) \( \{ z \in \mathbb{C} : z \text{ is real and } 0 \leq z \leq 1 \} \) is closed, since both endpoints are included. ■
#3: If \((X, d)\) is any metric space, show that every open ball is, in fact, an open set. Also, show that every closed ball is a closed set.

Let \(x\) and \(r\) be fixed. Consider \(B(x, r) = \{y \in X : d(x, y) < r\}\).

Suppose that \(y \in B(x, r)\). We want to show that there exists \(\epsilon > 0\) such that \(B(y, \epsilon) \subseteq B(x, r)\).

Since \(y \in B(x, r)\), there exists a positive number \(\epsilon\) such that \(d(x, y) = r - 2\epsilon\). For all points \(s\) such that \(d(y, s) < \epsilon\), we have that \(d(x, s) \leq d(x, y) + d(y, s) < r - 2\epsilon + \epsilon = r - \epsilon < r\), so we have that \(s \in B(x, r)\). Thus, \(B(x, r)\) is open.

A pictorial description appears to the right. ■

Let \(x\) and \(r\) again be fixed. Consider \(\overline{B}(x, r) = \{y \in X : d(x, y) \leq r\}\). To show that \(\overline{B}(x, r)\) is a closed set, we need to show that \(\overline{B}(x, r)^C = \{y \in X : d(x, y) > r\}\) is open.

That is, we want to show that there exists \(\epsilon > 0\) such that \(B(y, \epsilon) \subseteq \overline{B}(x, r)^C\).

Suppose that \(y \in \overline{B}(x, r)^C\). Thus, there exists \(\epsilon > 0\) such that \(d(x, y) = r + 2\epsilon\). By the triangle inequality, we have that \(d(x, y) \leq d(x, s) + d(s, y)\), i.e. \(d(x, s) \geq d(x, y) - d(y, s)\). For all points \(s\) such that \(d(y, s) < \epsilon\), we have that \(d(x, s) \geq d(x, y) - d(y, s) = r + 2\epsilon - \epsilon = r + \epsilon > r\), so we have that \(s \in \overline{B}(x, r)^C\). Thus, \(\overline{B}(x, r)^C\) is closed.

Again, a pictorial description appears to the right. ■

1.11 Proposition. Let \((X, d)\) be a metric space. Then:

(a) The sets \(X\) and \(\emptyset\) are closed;

(b) If \(F_1, \ldots, F_n\) are closed sets in \(X\), then so is \(\bigcup_{k=1}^n F_k\);

(c) If \(\{F_j : j \in J\}\) is any collection of closed sets in \(X\), \(J\) any indexing set, then \(\bigcap_{j \in J} F_j\) is also closed.

#5: Prove Proposition 1.11.

We begin with the following, useful lemma.
Lemma 3: Let $\{F_k\}$ be a (finite or infinite) collection of sets $F_k$. Then

(i) $\left(\bigcup_{k} F_k\right)^C = \bigcap_{k} (F_k)^C$ and (ii) $\left(\bigcap_{k} F_k\right)^C = \bigcup_{k} (F_k)^C$

Proof: If $x \in \left(\bigcup_{k} F_k\right)^C$, then $x \notin \bigcup_{k} F_k$, hence $x \notin F_k$ for any $k$, hence $x \in (F_k)^C$ for every $k$, so that $x \in \bigcap_{k} (F_k)^C$. Thus, we have that $\left(\bigcup_{k} F_k\right)^C \subseteq \bigcap_{k} (F_k)^C$.

Conversely, if $x \in \bigcap_{k} (F_k)^C$, then $x \notin (F_k)^C$ for every $k$, hence $x \notin F_k$ for any $k$, hence $x \notin \bigcup_{k} F_k$, so that $x \in \left(\bigcup_{k} F_k\right)^C$. Thus, $\left(\bigcup_{k} F_k\right)^C \supseteq \bigcap_{k} (F_k)^C$.

It follows that $\left(\bigcup_{k} F_k\right)^C = \bigcap_{k} (F_k)^C$, which completes the proof of (i). ■

To prove (ii), we take the complement of both sides of (i) and replace $F_k$ with $(F_k)^C$. ■

(a) Recall, a set is closed if its complement is open. Since $X^C = \emptyset$ and $\emptyset^C = X$, each of which is open, we have that both $X$ and $\emptyset$ are closed. ■

(b) By Lemma 3(i), $\left(\bigcup_{k} F_k\right)^C = \bigcap_{k} (F_k)^C$ and $(F_k)^C$ are open, since $F_k$ are closed.

Thus, Proposition 1.9(b) implies that $\left(\bigcup_{k=1}^{n} F_k\right)^C = \bigcap_{k=1}^{n} (F_k)^C$ is open as well. Thus, we have that $\bigcup_{k=1}^{n} F_k$ is closed, as desired. ■

(c) By Lemma 3(ii), $\left(\bigcap_{k} F_k\right)^C = \bigcup_{k} (F_k)^C$ and $(F_k)^C$ are open, since $F_k$ are closed.

Thus, Proposition 1.9(c) implies that $\left(\bigcap_{k} F_k\right)^C = \bigcup_{k} (F_k)^C$ is open as well. Thus, we have that $\bigcap_{k} F_k$ is closed, as desired. ■
I.6.7 \[ d(z, z') = \frac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}} \]

I.6.8 \[ d(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}} \]

#7: Show that \((\mathbb{C}_\infty, d)\) where \(d\) is given by (I.6.7) and (I.6.8) is a metric space.

To show that \((\mathbb{C}_\infty, d)\) is a metric space, we need to show that \(d\) satisfies:

(i) \(d(z, z') \geq 0\) and \(d(z, z') = 0\) if and only if \(z = z'\). 
\(d(z, \infty) \geq 0\) and \(d(z, \infty) = 0\) if and only if \(z = \infty\).

Notice that \(d(z, z') \geq 0\) from its construction. \(d(z, z') = 0\) iff \(|z - z'| = 0\), which happens if and only if \(z = z'\).

Similarly, \(d(z, \infty) \geq 0\) and if we consider the limit as \(z \rightarrow \infty\), we see that \(d(z, \infty) = 0\) if and only if \(z = \infty\), since the denominator heads towards infinity. ■

(ii) \(d(z, z') = d(z', z)\) and \(d(z, \infty) = d(\infty, z)\).

\[ d(z, z') = \frac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}} = \frac{2|-(z' - z)|}{\sqrt{(1 + |z'|^2)(1 + |z|^2)}} = \frac{2|z' - z|}{\sqrt{(1 + |z'|^2)(1 + |z|^2)}} = d(z', z). \]

And we have that \(d(z, \infty) = d(\infty, z)\) by its construction. ■

(iii) \(d(z, z'') \leq d(z, z') + d(z', z'')\),
\(d(z, z'') \leq d(z, \infty) + d(\infty, z'')\),
\(d(z, \infty) \leq d(z, z') + d(z', \infty)\).

These follow immediately from the fact that this metric represents the Euclidean distance between the stereographic images on the Riemann sphere and the triangle inequality holds for Euclidean distance. ■
#3: Which of the following subsets \( X \) of \( \mathbb{C} \) are connected: if \( X \) is not connected, what are its components? (a) \( X = \{ z : |z| \leq 1 \} \cup \{ z : |z - 2| < 1 \} \), (b) \( X = [0, 1) \cup \left\{ 1 + \frac{1}{n} : n \geq 1 \right\} \), (c) \( X = \mathbb{C} - (A \cup B) \), where \( A = [0, \infty) \) and \( B = \{ z = r \text{cis} \theta : r = \theta, 0 \leq \theta \leq \infty \} \)?

(a) \( |z| \leq 1 \) defines a unit disk centered at the origin while \( |z - 2| < 1 \) defined a unit disk (without its perimeter) centered at \((2, 0)\). The subset is connected via the point \((1, 0)\). Thus, we have that \( X = \{ z : |z| \leq 1 \} \cup \{ z : |z - 2| < 1 \} \) is connected. ■

(b) \( X = [0, 1) \cup \left\{ 1 + \frac{1}{n} : n \geq 1 \right\} \) is not connected. \( \lim_{n \to \infty} 1 + \frac{1}{n} = 1 \not\in [0,1) \). If the two pieces were to be connected, they would meet at the point \( x = 1 \). ■

(c) \( X = \mathbb{C} - (A \cup B) \) is not going to be connected. Notice that \( X = \{ \text{shaded region} \} \). If we only considered \( \mathbb{C} - B \), then this would be connected. The subtraction of half of the real axis ensures that we will have a set of spirals, none of which are connected to each other. ■
#4: Let \( z_n, z \) be points in \( \mathbb{C} \) and let \( d \) be the metric on \( \mathbb{C}_\infty \). Show that \( |z_n - z| \to 0 \) if and only if \( d(z_n, z) \to 0 \). Also show that if \( |z_n| \to \infty \), then \( \{z_n\} \) is Cauchy in \( \mathbb{C}_\infty \). (Must \( \{z_n\} \) converge in \( \mathbb{C}_\infty \)?)

Looking at I.6.7, it is clear that if \( |z_n - z| \to 0 \), then \( d(z_n, z) \to 0 \). Similarly, for I.6.8, if \( z = \infty \), then if \( |z_n - \infty| \to 0 \), then \( d(z_n, \infty) \to 0 \).

In I.6.7, suppose that \( |z_n - z| \not\to 0 \). We want to show that \( d(z_n, z) \not\to 0 \) as well. There are two cases to consider: (i) \( |z_n - z| \to \infty \) (which necessarily implies that \( |z_n| \to \infty \)) and (ii) \( |z_n - z| \to L \), where \( 0 < L < \infty \) (which implies that \( |z_n| \) is finite).

For (i), \( d(z_n, z) = \frac{2|z_n - z|}{\sqrt{(1 + |z_n|^2)(1 + |z|^2)}} \geq \frac{2|z_n| - 2|z|}{1/|z_n|} = \frac{2 - 2(|z|/|z_n|)}{\sqrt{\left(1/|z_n|^2 + 1\right)(1 + |z|^2)}} \). As \( n \to \infty \), we have that \( \frac{2 - 2(|z|/|z_n|)}{\sqrt{\left(1/|z_n|^2 + 1\right)(1 + |z|^2)}} \to \frac{2}{\sqrt{(1 + |z|^2)}} > 0 \), so we have that \( d(z_n, z) \not\to 0 \). For (ii), since \( |z_n - z| \not\to 0 \), it suffices to verify that the denominator \( \sqrt{(1 + |z_n|^2)(1 + |z|^2)} < \infty \). However, since \( z \) is fixed and \( |z_n| \) is finite, we are done.

In I.6.8, notice that if \( d(z_n, \infty) \to 0 \), then it must be that \( z_n \to \infty \), i.e. \( |z_n - \infty| \to 0 \).

Now, suppose \( |z_n| \to \infty \). We want to show that for \( n, m \) sufficiently large, then we have that \( d(z_n, z_m) < \epsilon \) for any \( \epsilon > 0 \). Let \( \epsilon > 0 \) be fixed. By the triangle inequality, \( d(z_n, z_m) \leq d(z_n, \infty) + d(\infty, z_m) = \frac{2}{\sqrt{1 + |z_n|^2}} + \frac{2}{\sqrt{1 + |z_m|^2}} \). Since \( |z_n| \to \infty \), there exists \( N_1 \) such that \( \frac{2}{\sqrt{1 + |z_n|^2}} < \frac{\epsilon}{2} \) for all \( n > N_1 \). Similarly, since \( |z_m| \to \infty \), there exists \( N_2 \) such that \( \frac{2}{\sqrt{1 + |z_m|^2}} < \frac{\epsilon}{2} \) for all \( m > N_2 \). Let \( N = \max \{N_1, N_2\} \). Then \( d(z_n, z_m) \leq \epsilon/2 + \epsilon/2 = \epsilon \) for all \( m, n > N \). \( \blacksquare \)
#6: Give three examples of non-complete metric spaces.

We want to find a metric space \((X, d)\) such that there exists a Cauchy sequence that does \textit{not} have a limit in \(X\).

1. Let \(X = \mathbb{Q}\), with the standard metric of absolute value. Consider the (Cauchy) sequence defined by \(x_1 = 1\), \(x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}\). The limit, \(x\), of the sequence would need to satisfy \(x^2 = 2\). No rational number solves this equation. ■

2. Let \(X = \mathbb{R}\setminus\{0\}\), again with the standard metric. Consider the (Cauchy) sequence \(\{1/n^2, \, n = 1, 2, \ldots\}\). The limit of this sequence is 0, which is not in \(X\). ■

3. Let \(X\) be the open interval \((0, 1)\), again with the standard metric. In a similar fashion as in (2), consider the (Cauchy) sequence \(\{1/n, \, n = 2, 3, \ldots\}\). The limit of this sequence is 0, which is not in the interval. ■

#7: Put a metric \(d\) on \(\mathbb{R}\) such that \(|x_n - x| \to 0\) if and only if \(d(x_n, x) \to 0\), but that \(\{x_n\}\) is a Cauchy sequence in \((\mathbb{R}, \, d)\) when \(|x_n| \to \infty\). (Hint: Take inspiration from \(C_{\infty}\).)

Let \(d(x_n, x) = \frac{2|x_n - x|}{\sqrt{(1 + |x_n|^2)(1 + |x|^2)}}\). The desired properties hold based on the results from Problem #4 (above). ■