INSTRUCTIONS

1. NO CALCULATOR.
2. CLOSE BOOK, CLOSE NOTES.
3. ID WILL BE CHECKED. GET IT READY!

General instructions: 3 hours. Be sure to carefully motivate all (nontrivial) claims and statements. You may use without proof any result proved in the text. If you use a theorem from the text, refer to it either by name (if it has one) or state what it says. Also verify explicitly all hypotheses in the theorem. You need to reprove any result given as an exercise.

Notation: $\chi_E$ denotes the characteristic function of a set $E$. $\int_a^b f(x)\,dx$ denotes the integral with respect to the Lebesgue measure on $(a,b)$. The Lebesgue measure itself is denoted $m$. 
1. Determine if the statement below are **True** or **False**. If **True**, give a brief proof. If **False**, give a counterexample (or prove your assertion in another way, if you prefer). If your claim follows from a theorem in the text, name the theorem (or describe it otherwise) and explain carefully how the conclusion follows.

a) If \( E \) is a Borel set of \( \mathbb{R}^n \). Define

\[
D_E(x) = \lim_{r \to 0} \frac{m(E \cap B(r, x))}{m(B(r, x))}
\]

whenever the limit exists. Then \( D_E(x) = 1 \) a.e. \( x \in E \).

**Solution:** (Exercise 25 of 3.4) True. Follows from the Lebesgue differentiation theorem which asserts that for any \( f \in L^1_{loc} \)

\[
\lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)dy = f(x)
\]

a.e. \( x \) applying to \( \chi_E(x) \), which is \( L^1 \)-locally.

b) Suppose \( \{\nu_j\} \) is a sequence of positive measures. If \( \nu_j \perp \mu \) for all \( j \), then \( \sum_{j=1}^{\infty} \nu_j \perp \mu \).

**Solution:** (Exercise 9 of 3.2) True. Let \( A_j \cup B_j \) be the decomposition for \( \nu_j \) and \( \mu \) with \( \nu_j \mid_{A_j} = 0 \) and \( \mu \mid_{B_j} = 0 \). Let \( A = \cap A_j \) and \( B = \cup B_j \). It can be checked that \( X = A \cup B \) and \( \sum \nu_j \mid_A = 0 \) and \( \mu \mid_B = 0 \).

c) If \( \mu^* \) is an outer measure on \( X \) and \( \{A_j\} \) is a sequence of disjoint \( \mu^* \)-measurable sets, then \( \mu^*(E \cap (\cup_{j=1}^{\infty} A_j)) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j) \) for any subset \( E \).

**Solution:** (Exercise 17 of 1.4) True. Details see the solution to the HW solution. Note that here \( E \) is not assumed to be measurable.

First by the definition \( \mu^*(E \cap (\cup_{j=1}^{\infty} A_j)) \leq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) \) On the other hand \( \mu^*(E \cap (\cup_{j=1}^{n} A_j)) = \mu^*(E \cap (\cup_{j=1}^{n} A_j) \cap A_n) + \mu^*(E \cap (\cup_{j=1}^{n} A_j) \cap A_n^c) \), which implies that

\[
\mu^*(E \cap (\cup_{j=1}^{n} A_j)) = \mu^*(E \cap A_n) + \mu^*(E \cap (\cup_{j=1}^{n-1} A_j)).
\]

Hence by the induction we have that \( \mu^*(E \cap (\cup_{j=1}^{\infty} A_j)) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j) \). This implies that \( \mu^*(E \cap (\cup_{j=1}^{\infty} A_j)) \geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) \).
d) Let \((X, \mathcal{M}, \mu)\) be a measure space. Assume that \(f_n \to f\) and \(g_n \to g\) in measure. Then \(f_ng_n \to fg\) in measure.

**Solution:** (Exercise 38 of 2.4) False. Let \(f_n = g_n = x + \frac{1}{n}\). They form a counter-example to the claim. Or \(f_n = x, g_n = \frac{1}{n}\).

e) The function \(f(x) = x \sin(x^{-1})\), for \(x \neq 0\), and \(f(0) = 0\), is a BV function on \([0,1]\).

**Solution:** (Example did in the lecture of 12/10/2014) False. Note that the function is continuous. If we chose \(x_j = \frac{2}{(2j+1)\pi}\) for \(1 \leq j \leq n\), the \(V_0^1(f)\) can be estimated from below by \(\frac{1}{\pi} \sum_{j=1}^{n} \frac{1}{2j+1}\). Since \(\sum_{j=1}^{n} \frac{1}{2j+1} \to \infty\) as \(n \to \infty\) we have that \(V_0^1(f) = \infty\).

f) Let \((X, \mathcal{M}, \mu)\) be a complete measure space. If \(f_n, g_n, g, f \in L^1\), \(f_n \to f\) and \(g_n \to g\) a.e., \(|f_n| \leq g_n\) and \(\int g_n = A < \infty\) for some \(A > 0\), then \(\int f_n \to \int f\).

**Solution:** False. Many counter examples. The easiest is to have \(f_n = g_n\) for a sequence the interchange of the limit and the integration fails. For example \(f_n = \frac{1}{n}\chi_{(0,n)} \to 0\), but \(\int f_n = 1\).
2. If $f$ is an increasing function on $\mathbb{R}$ prove that $f(b) - f(a) \geq \int_a^b f'(t) \, dt.$

Solution: (Exercise 33 of 3.5) I did this problem in the lecture of December 8th, 2014.

Solution 1: let $F(x) = f(b)$ for $x \geq b$ and $F(x) = f(a)$ for $x \leq a$. $F(x)$ clearly is monotone and we only need to prove the result for $F$.

$$\frac{1}{h} \int_a^b F(t + h) - F(t) = \frac{1}{h} \left( \int_b^{b+h} F(t) - \int_a^{a+h} F(t) \right) \geq F(b) - F(a).$$

Taking limit $h \to 0$, the result follows by applying Fatou's lemma.

Solution 2: Let $g(x) = F(x+).$ Then by a result in the book (Theorem 3.23) we have that $g'(x) = f'(x)$ a.e. $x \in [a, b]$. Clearly $g(b) - g(a) \geq f(b) - f(a)$. Let $\mu_g$ be the Lebesgue-Stieltjes measure defined by $g$, which is a Borel measure and regular. By L-R-N theorem and Lebesgue differentiation theorem we have that $d\mu_g = d\lambda + g' dm$ where $\lambda \perp m$. Hence

$$g(b) - g(a) = \mu_g((a, b]) \geq \int_a^b g' = \int_a^b f'.$$
3. Suppose that \(\mu, \nu\) are \(\sigma\)-finite (positive) measures on \((X, \mathcal{M})\) with \(\nu \ll \mu\) and let \(\lambda = \nu + \mu\). If \(f = \frac{d\nu}{d\lambda}\), then \(0 \leq f < 1\), \(\mu\) a.e. and \(\frac{d\nu}{d\mu} = \frac{f}{1-f}\).

\[
\text{Solution: (Exercise 16 of 3.2)}
\]

Let \(g = \frac{d\nu}{d\mu}\). It is easy to see that \(\nu \ll \lambda\) and \(\mu \ll \lambda\) by the definitions. By the definition \(f \geq 0\ \lambda\) a.e. which implies that \(f \geq 0\ \mu\) a.e.

For \(f < 1\), let \(E = \{x | f(x) \geq 1\}\). Then

\[
\nu(E) = \int_E f \, d\lambda \geq \lambda(E) = \nu(E) + \mu(E)
\]

which implies that \(\mu(E) = 0\).

Finally

\[
1 = \frac{d\lambda}{d\lambda} = \frac{d\nu}{d\lambda} + \frac{d\mu}{d\lambda} = \left( \frac{d\nu}{d\mu} + 1 \right) \frac{d\mu}{d\lambda}
\]

But clearly \(\frac{d\nu}{d\lambda} = 1 - f\). Hence we have \(\frac{d\nu}{d\mu} = \frac{1}{1-f} - 1 = \frac{f}{1-f}\).
4. Assume that $f$ is a bounded function on $\mathbb{R}$. If for any $x$, $\lim_{h \to 0, h \neq 0} f(x + h)$ exists. Prove that $f$ is Riemann integrable on any bounded interval $[a, b]$.

Solution:

Recall the theorem from the book (proved in the lecture carefully):

If $f$ is a bounded function on $[a, b]$. Then $f$ is Riemann integrable if and only if that \( \{ x \mid f \text{ is discontinuous at } x \} \) is of Lebesgue measure zero.

Remark: Very few even quoted this result accurately.

Now we claim that for any $[a, b]$, a finite closed interval, there are only finite many point satisfying that

$$\lim_{h \to 0, h \neq 0} |f(x + h) - f(x)| \geq \frac{1}{k}$$

for any integer $k > 0$. The existence of the limit is ensured by the assumption. We denote such set by $E_k$. Clearly this would be sufficient by the above result. (Some people making uses of all kinds of result on Lebesgue integrable functions. Since a priori (before you prove it) we do not even know $f$ is (Lebesgue) measurable, all those applications have absolute no ground.)

We prove the claim by contradiction. Assume not true, namely there are $x_n \in [a, b] \cap E_k$ such that $x_n$ distinct and $x_n$ are discontinuous point of $f$. By the assumption that $\lim_{h \to 0} f(x + h)$ exists we have that $f(x_n) \neq \lim_{h \to 0} f(x_n + h)$. By the compactness of $[a, b]$, there exists subsequence of $\{x_n\}$ (still denote by $\{x_n\}$) so that it converges/accumulates to some point $x_0 \in [a, b]$ with $x_n \neq x_0$. Since $x_n \to x_0$ we have that

$$\lim_{n \to \infty} f(x_n) = \lim_{h \to 0} f(x_0 + h)$$

One the other hand, by definition of $E_k$, we can find $h_n$ such that $|h_n| \leq 1/2^n$ such that $x_n + h_n \neq x_0$ and

$$|f(x_n + h_n) - f(x_n)| \geq \frac{1}{k}.$$ 

On the other hand since $x_n + h_n \to x_0$ we also have that $\lim_{n \to \infty} f(x_n + h_n) = \lim_{h \to 0} f(x_0 + h)$. This is a contradiction.
5. Suppose that $(X, \mathcal{M}, \mu)$ is a σ-finite measure space and $f \in L^+(X)$. Let

$$G_f = \{(x, y) \in X \times [0, \infty] : y \leq f(x)\}.$$ 

Prove (i) $G_f$ is measurable. (ii) $(\mu \times m)(G_f) = \int f \, d\mu$.

Solution: (Exercise 50 of 2.5) (ii) is a simple application of Tonelli’s theorem to $L^+$ functions.