1 Notation

In the problems below,

- \( G \) represents a graph on \( n \) vertices, with vertex set \( V \) and edge set \( E \).
- \( A \) represents the adjacency matrix of \( G \).
- \( D \) represents the diagonal degree matrix of \( G \).
- \( L \) represents the combinatorial Laplacian of \( G \), so \( L = D - A \).
- \( \mathcal{L} \) represents the (normalized) Laplacian of \( G \), so \( \mathcal{L} = I - D^{-1/2}AD^{-1/2} \).

Problem 1. Find examples of families of graphs with different eigenvectors for \( A \), \( L \), and \( \mathcal{L} \) (that is, show that the three spectra are fundamentally different).

Problem 2 (*). Find two nonisomorphic graphs that are cospectral with respect to \( A \). Find two nonisomorphic graphs that are cospectral with respect to \( L \).

31 March 2011

Here we define \( P = D^{-1}A \), the probability transition matrix for a random walk on \( G \), and \( \pi \) to be the stationary distribution of such a walk. We further define the following three measures of the distance between \( P^t \) and the stationary distribution \( \pi \):

\[
\Delta_{TV}(t) = \max_{u \in V} \max_{A \subseteq V} \sum_{v \in A} \left| \sum_{u \in V} \chi_u P^t(v) - \pi(v) \right|,
\]

\[
\Delta(t) = \max_{u, v \in V} \frac{|\chi_u P^t(v) - \pi(v)|}{\pi(v)},
\]

\[
\Delta_{\chi}(t) = \max_{u} \left( \sum_{v \in V} \frac{(\chi_u P^t(v) - \pi(v))^2}{\pi(v)} \right)^{1/2}.
\]

Problem 1. Prove that

\[
\Delta_{TV}(t) = \frac{1}{2} \max_{u \in V} \sum_{v \in V} |\chi_u P^t(v) - \pi(v)|.
\]

Problem 2. Prove that \( \Delta(t) \geq \Delta_{\chi}(t) \).
Problem 3. Complete the proof (begun in class) that
\[ \Delta(t) \leq \max_{u,v \in V} \frac{\bar{\lambda} \| \chi_u D^{-1/2} \| \| \chi_v D^{1/2} \|}{d_v \text{Vol}(G)}, \]
where \( 0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1} \) are the eigenvalues of \( \mathcal{L} \) and
\[ \bar{\lambda} = \max_{i \neq 0} |1 - \lambda_i|. \] (1)

Problem 4. Prove that the eigenvector corresponding to the eigenvalue 0 of \( \mathcal{L} \) is given by
\[ \varphi_0(v) = \sqrt{d_v / \text{Vol}(G)}. \]

5 April 2011
Here we define
\[ \| P^t \|_r = \max_u \left( \sum_v \pi(v) \left( \frac{P^t(u,v) - \pi(v)}{\pi(v)} \right)^r \right)^{1/r}. \]
Notice, then, that using the definitions given in the previous lecture, \( \| P^t \|_1 = 2\Delta TV(t), \| P^t \|_2 = \Delta \chi(t), \) and \( \| P^t \|_\infty = \Delta(t). \)

Problem 1. Prove that if \( r \leq s, \| P^t \|_r \leq \| P^t \|_s. \)

Problem 2. Given \( \epsilon > 0 \) and \( \bar{\lambda} \) as in (??), prove that if \( \| P^t \|_1 < \epsilon, \) then \( t > \frac{1}{ \lambda_{\text{min}} \log(1/\epsilon)}. \)

7 April 2011
Problem 1. Use the Rayleigh quotient to prove the following facts about the spectrum of \( \mathcal{L} \):
   1. \( \lambda_{n-1} \leq 2 \) (Hint: use the fact that \( (a - b)^2 \leq 2(a^2 + b^2) \))
   2. \( G \) is bipartite if and only if \( \lambda_{n-1} = 2 \)
   3. If \( G \) is bipartite, and \( \lambda_i \) is an eigenvalue, then \( 2 - \lambda_i \) is also an eigenvalue of equal multiplicity.

Problem 2. Find the spectrum of \( C_n \) and a corresponding orthonormal eigenbasis. More generally, find the eigenvalues of any symmetric cyclic matrix.

Problem 3 (*). If \( D \) is the diameter of a \( k \)-regular graph \( G \), prove that \( \lambda_1 \leq 1 - \frac{2\sqrt{k-1}}{k} (1 - \frac{2}{D}) + \frac{2}{D}. \)

Problem 4 (**). Develop a bound for \( \lambda_1 \), similar to the Alon-Boppana bound (above) for general graphs.