Solutions to Homework set 2, Math 180B

Ch I, Ex 5.5: Let service rate for clerk $i$ be $\lambda_i$, $i = 1, 2$. It’s assumed that all service times are independent, exponential. Let $T_1, T_2, T_3$ denote the respective service times for John, Paul, Naomi.

(a) It’s assumed here that $\lambda_1 = \lambda_2 = \lambda$. As in the discussion in section 5.2, let $U := \min(T_1, T_2)$ denote the time at which the first person leaves, let $V := \max(T_1, T_2)$ denote the time when both John and Paul have left, and let $W := V - U$. Then Naomi leaves at time $T = U + T_3$. We are asked for $P\{T > V\} = P\{U + T_3 > U + W\} = P\{T_3 > W\}$. However, $T_3$ is independent of $T_1$ and $T_2$, hence of $W$. The distribution of $W$ was computed in (5.2) as

$$P\{W > t\} = \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-\lambda_2 t} + \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_1 t}.$$ 

Setting $\lambda_1 = \lambda_2 = \lambda$, this reduces to $P\{W > t\} = e^{-\lambda t}$. Therefore $T_3$ and $W$ are independent with the same density, hence $P\{T_3 > W\} = 1/2$.

(b) We now assume different service rates. The expression above for $P\{W > t\}$ gives, on differentiation with respect to $t$,

$$f_W(t) = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_2 t} + \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_1 t} = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} (e^{-\lambda_1 t} + e^{-\lambda_2 t}).$$ 

As in the discussion in 5.2, let $N$ be the indicator of the event $T_2 \leq T_1$, which is to say, Paul leaves before John. The distribution of $T_3$ is given by

$$P\{T_3 > t\} = P\{T_3 > t \mid N = 1\}P\{N = 1\} + P\{T_3 > t \mid N = 0\}P\{N = 0\}.$$ 

If $N = 1$ (an event with probability $\lambda_2/(\lambda_1 + \lambda_2)$, by 5.2 (a)), then Naomi is served by the second server, so $P\{T_3 > t \mid N = 1\} = e^{-\lambda_2 t}$, while on the other hand, $P\{T_3 > t \mid N = 0\} = e^{-\lambda_1 t}$. Therefore we have

$$P\{T_3 > t\} = e^{-\lambda_2 t} \frac{\lambda_2}{\lambda_1 + \lambda_2} + e^{-\lambda_1 t} \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$ 

Using independence of $T_3$ and $W$ and the distributions of $T_3$ and $W$ determined above, we find

$$P\{T_3 > W\} = \int_0^\infty \int_0^\infty 1_{\{t > w\}} f_{T_3}(t) f_W(w) \, dw \, dt.$$ 

If we integrate first over $t$ from $w$ to $\infty$, this reduces to

$$P\{T_3 > W\} = \int_0^\infty (e^{-\lambda_2 w} \lambda_2 \lambda_1 + \lambda_2 + e^{-\lambda_1 w} \lambda_1 \lambda_1 + \lambda_2) f_W(w) \, dw.$$ 

Substituting the expression above for $f_W(w)$ gives

$$P\{T_3 > W\} = \frac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)^2} \int_0^\infty (e^{-\lambda_2 w} \lambda_2 + e^{-\lambda_1 w} \lambda_1)(e^{-\lambda_1 w} + e^{-\lambda_2 w}) \, dw = \frac{2\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)^2}.$$ 

If we set $\lambda_1 = 3$ and $\lambda_2 = 47$, this reduces to $P\{T_3 > W\} = 2 * 3 * 47/50^2 = 282/2500 \approx .11$.

Chap II, Ex. 1.5: Let $A$ denote the event \{ $X \in \{1, 3, 5, \ldots \}$ \}. Then we seek $E\{X \mid A\} := E\{X_1A\}/P\{A\}$. But

$$E\{X_1A\} = \sum_{k=0}^\infty E\{X_1A \mid X = k\} P\{X = k\} = \sum_{j=0}^\infty E\{X_1A \mid X = 2j + 1\} P\{X = 2j + 1\}.$$ 

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Substituting Poisson probabilities in this gives
\[ \sum_{j=0}^{\infty} (2j + 1)e^{-\lambda} \frac{\lambda^{2j+1}}{(2j+1)!} = \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^{2j}}{(2j)!} = \frac{\lambda e^{-\lambda} e^\lambda}{2} = \frac{\lambda + e^{-2\lambda}}{2}. \]

On the other hand,
\[ P(A) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^{2k+1}}{(2k+1)!} = e^{-\lambda} \sinh(\lambda) = \frac{1 - e^{-2\lambda}}{2} \]

and so
\[ E\{X | A\} = \frac{1 + e^{-2\lambda}}{1 - e^{-2\lambda}}. \]

**Chap II, Prob. 1.7:** Let \( A \) denote “accident due to structural failure”, let \( B \) denote “accident diagnosed as structural failure”. We are given:
\[ P(A) = .3 \]
\[ P(A^c) = 1 - P(A) = .7 \]
\[ P(B | A) = .85 \]
\[ P(B | A^c) = .35. \]
We seek \( P(A | B) \). (This is actually a Bayes’ formula problem, but we’ll do it from scratch.)
\[ P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B | A)P(A)}{P(B | A)P(A) + P(B | A^c)P(A^c)} = \frac{.85 \times .3}{.85 \times .3 + .35 \times .7} = .51. \]

**Chap II, Prob. 1.10:** Let \( X \) denote the number of children in a family \((1 \leq X \leq 3)\), \( Y \) the number of boys \((0 \leq Y \leq 3)\), \( Z \) the number of girls, so \( X = Y + Z \), and \( 0 \leq Z \leq 1 \).
\( a) \ P\{X = 1\} = P\{\text{first child is girl}\} = \frac{1}{2}. \)
\( P\{X = 2\} = P\{\text{first child is boy, second is girl}\} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}. \)
\( P\{X = 3\} = P\{\text{first child is boy, second is boy, third is boy or girl}\} = \frac{1}{2} \times \frac{1}{2} \times 1 = \frac{1}{4}. \)
\( b) \ As \ Z \ takes \ only \ the \ values \ 0, 1 \)
\( P\{Z = 0\} = P\{3 \text{ boy children}\} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}. \)
\( P\{Z = 1\} = 1 - P\{Z = 0\} = 7/8. \)
\( c) \ The \ probabilities \ computed \ in \ (a) \ and \ (b) \ assume \ in \ effect \ that \ a \ family \ was \ selected \ at \ random. \)
Imagine a population with \( N \) families \((N \) large). According to the analysis in \( a) \), those families will have no boys with probability \( 1/2 \), 1 boy with probability \( 1/4 \), 2 boys with probability \( 1/8 \) and 3 boys with probability \( 1/8 \). Therefore, we may suppose that the number of boys in the population is \( N/4 + 2 \times N/8 + 3 \times N/8 = 7N/8 \). If we select a boy at random, the chance that he is from a family with respectively 1, 2, 3 boys is \( \frac{N/4}{7N/8}, \frac{2N/8}{7N/8}, \frac{3N/8}{7N/8} \). Therefore, the distribution of the number of boy siblings is given by probabilities \( 2/7, 2/7, 3/7 \) of respectively 0, 1, 2 boy siblings. The event “no girl siblings” can occur only if the family consists of 3 boys, so the probability of no girl siblings is \( \frac{3N/8}{7N/8} = 3/7 \), and the probability of one girl sibling (the complement) is \( 4/7 \).

**Chap II, Ex. 2.1:** Let \( X \) denote the result of the red die toss, \( Y_1, Y_2, \ldots \) the results of repeatedly tossing the green die. Let \( N \) denote the first index \( k \) for which \( X + Y_k \) is either 4 or 7. We seek \( P\{X + Y_N = 4\} \). By the formula for total probability,
\[ P\{X + Y_N = 4\} = \sum_{j=1}^{6} P\{X + Y_N = 4 | X = j\} P\{X = j\} = \frac{1}{6} \sum_{j=1}^{3} P\{Y_N = 4 - j | X = j\}. \]
But, conditional on $X = j$ $(1 \leq j \leq 3)$, $Y_N$ is equally likely to be $4 - j$ and $7 - j$, and therefore

$$P\{X + Y_N = 4\} = \frac{1}{6} \sum_{i=1}^{3} \frac{1}{2} = \frac{1}{4}.$$  

**Chap II, Ex. 3.1**: Let $X_1, X_2, \ldots$ denote the indicators of heads in a sequence of independent coin tosses—that is, Bernoulli trials with probability $1/2$ of success. Let $S_n = X_1 + \ldots + X_n$, so that $S_n$ is Binomial$(n, p)$ and $Z = S_N$. That is, $Z$ is a random sum of $N$ independent Bernoulli variables, with possible values in the range 0 to 6. Using the formulas for the moments in section give

$$EZ = ENEX_1; \quad VarZ = ENVa r(X_1) + Var(N)(EX_1)^2$$

But, $N$ is uniform on $\{1, \ldots , 6\}$, and so $EN = 7/2$ and $Var(N) = 35/12$, while $EX_1 = 1/2$, $EX_1^2 = 1/2$, $Var(X_1) = 1/4$. Plugging these in gives

$$EZ = \frac{7}{2} \cdot \frac{1}{2} = \frac{7}{4}; \quad Var(Z) = \frac{7}{2} \cdot \frac{1}{4} + \frac{35}{12} \cdot \frac{1}{4} = \frac{77}{48}.$$ 

The probability mass function of $Z$ is, for $0 \leq k \leq 6$,

$$p_Z(k) = P\{Z = k\} = \sum_{n=1}^{6} P\{Z = k \mid N = n\}P\{N = n\} = \frac{1}{6} \sum_{n=1}^{6} P\{Z = k \mid N = n\}.$$ 

Conditional on $N = n$, $Z$ is Binomial$(n, 1/2)$, giving

$$p_Z(k) = \frac{1}{6} \sum_{n=1}^{6} \left(\frac{n}{k}\right)^{1/2} \sum_{n=1}^{6} \left(\frac{n}{k}\right)^{1/2}.$$ 

The respective numeric values here are (with a little computer help)

$$\{21/128, 5/16, 33/128, 1/6, 29/384, 1/48, 1/384\}$$

From which we find first moment $\mu_1 = \sum_{k=0}^{6} k p_N(k) = 7/4$, second moment $\mu_2 = \sum_{k=0}^{6} k^2 p_N(k) = 14/3$, whence $EZ = 7/4$ and $Var(Z) = 14/3 - (7/4)^2 = 77/48 \approx 1.604$.

**Chap II, Ex. 3.2**: Let $X_1, X_2, \ldots$ denote the indicators of tails in a sequence of independent tosses of a dime—that is, Bernoulli trials with probability $1/2$ of success. Let $S_n = X_1 + \ldots + X_n$, so that $S_n$ is Binomial$(n, 1/2)$ and $Z = S_N$. As $N$ is Binomial$(6, 1/2)$, we have $EN = 6 \cdot 1/2 = 3$, $Var(N) = 6 \cdot 1/2 \cdot 1/2 = 3/2$. By the formulas in section 3.2, $EZ = ENEX_1$ and $VarZ = ENVa r(X_1) + Var(N)(EX_1)^2$. As $EX_1 = EX_1^2 = 1/2$, and hence $Var(X_1) = 1/2 - (1/2)^2 = 1/4$, and this leads to

$$EZ = 3 \cdot 1/2 = 3/2; \quad Var(Z) = 3 \cdot 1/4 + 3/2 \cdot (1/2)^2 = 9/8.$$ 

We compute $P\{Z = 2\}$ from the formula $P\{Z = 2\} = \sum_{k=0}^{6} P\{Z = 2 \mid N = k\}P\{N = k\}$. Substitute the binomial $(6, 1/2)$ probabilities for $N$ and use the fact that $P\{Z = 2 \mid N = k\} = P\{S_k = 2\} = \binom{k}{2} 2^{-k}$ (which vanishes if $k < 2$) to get

$$P\{Z = 2\} = \sum_{k=2}^{6} \binom{k}{2} 2^{-k} \binom{6}{k} 2^{-6} = \frac{1215}{4096} \approx .297.$$ 

**Chap II, Ex. 3.5**: Let $N$ denote the number of accidents in a week. Let $X_1, X_2, \ldots$ denote independent random variables representing the numbers of individuals injured on successive accidents, $S_n = X_1 + \ldots + X_n$. Then the number of individuals injured in a week is $W = S_N$. By the formulas in section 3.2, $EW = ENEX_1$ and $VarW = ENVa r(X_1) + Var(N)(EX_1)^2$. But $EN = 2 = Var(N)$ since $N$ is Poisson$(2)$, and we are given $EX_1 = 3$, $Var(X_1) = 4$. Therefore

$$EW = 2 \cdot 3 = 6; \quad Var(W) = 2 \cdot 4 + 2 \cdot 3^2 = 26.$$