Lévy’s Theorem

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space endowed with a right-continuous* filtration \((\mathcal{F}_t)_{t \geq 0}\) such that \(\mathcal{F}_0\) contains all the \(\mathbb{P}\)-null sets in \(\mathcal{F}\) and \(\mathcal{F}_t = \mathcal{F}\). Let \(M = (M_t)_{t \geq 0}\) be a real-valued stochastic process adapted to \((\mathcal{F}_t)\) with continuous sample paths. We assume that \(M_0 = 0\).

Theorem. Suppose that both \(M\) and \((M_t^2 - t)_{t \geq 0}\) are local martingales. Then \(M\) is a Brownian motion with respect to \((\mathcal{F}_t)\). More precisely, if \(0 < s < t\), then \(M_t - M_s\) is independent of \(\mathcal{F}_s\) and is normally distributed with mean \(0\) and variance \(t - s\).

Proof. The key observation (due to H. Kunita & S. Watanabe) is that the development of the Itô integral (and Itô’s formula) for Brownian motion \(B_t\) rests solely on the fact that \(B_t\) and \(B_t^2 - t\) are (local) martingales. It follows that if \(f \in C^2(\mathbb{R})\) then

\[
(1) \quad f(M_t) = f(0) + \int_0^t f'(M_s) \, dM_s + \frac{1}{2} \int_0^t f''(M_s) \, ds,
\]

where the stochastic integral \(M_t^f := \int_0^t f'(M_s) \, dM_s\) is a local martingale. In particular, if \(f\) and its derivatives \(f'\) and \(f''\) are bounded, then \(M_t^f\) is a martingale, in which case upon taking expectations in (1) we obtain

\[
(2) \quad \mathbb{E}[f(M_t)] = f(0) + \frac{1}{2} \int_0^t \mathbb{E}[f''(M_s)] \, ds.
\]

Let us take \(f\) in (2) to be of the form \(f(x) = \exp(i\theta x)\), where \(\theta \in \mathbb{R}\) and \(i = \sqrt{-1}\). Writing \(g(t) := \mathbb{E}[\exp(i\theta M_t)]\) we obtain

\[
g(t) = 1 - \frac{\theta^2}{2} \int_0^t g(s) \, ds
\]

because \(f''(x) = -\theta^2 f(x)\). Consequently, \(g\) satisfies the initial value problem

\[
g'(t) = -\frac{\theta^2}{2} g(t) \quad g(0) = 1,
\]

which has the unique solution \(g(t) = \exp(-t\theta^2/2)\). Thus

\[
\mathbb{E}[\exp(i\theta M_t)] = \exp(-t\theta^2/2), \quad \theta \in \mathbb{R},
\]

* i.e., \(\mathcal{F}_t = \mathcal{F}_{t+}\) for all \(t \geq 0\).
which means that $M_t \sim \mathcal{N}(0,t)$.

Now fix $s > 0$ and $A \in \mathcal{F}_s$ with $P(A) > 0$. Define $P^*(B) := P(B \cap A) / P(A) = P(B|A)$, $\mathcal{F}^*_t := \mathcal{F}_{t+s}$, and $M^*_t := M_{t+s} - M_s$ for $t \geq 0$. Then with respect to the filtration $(\mathcal{F}^*_t)$ over the probability space $(\Omega, \mathcal{F}, P^*)$, the stochastic process $(M^*_t)_{t \geq 0}$ is a continuous local martingale with $M^*_0 = 0$ such that $[M^*_t]^2 - t$ is also a local martingale.

The considerations of the preceding paragraph apply to this process, and we deduce that

$$(3) \quad E^*[\exp(i\theta M^*_t)]=\exp(-t\theta^2/2).$$

Writing the “starred” objects explicitly, (3) becomes

$$(4) \quad E[\exp(i\theta(M_{t+s} - M_t)); A] = \exp(-t\theta^2/2)P(A).$$

Varying $A \in \mathcal{F}_s$ in (4) we find that

$$E[\exp(i\theta(M_{t+s} - M_t))|\mathcal{F}_s] = \exp(-t\theta^2/2),$$

which shows that $M_{t+s} - M_s$ is independent of $\mathcal{F}_s$ and has the $\mathcal{N}(0,t)$ distribution. □

**Example 1.** Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space, and let $B = (B_t)_{t \geq 0}$ be an $(\mathcal{F}_t)$ Brownian motion. Let $H = (H_t)_{t \geq 0}$ be a measurable $(\mathcal{F}_t)$ adapted process taking on only the two values $\pm 1$. Then $H \in L^2$ so the stochastic integral

$$M_t := \int_0^t H_s dB_s, \quad t \geq 0,$$

is a square-integrable martingale. Moreover, $\langle M \rangle_t = \int_0^t H_s^2 \, ds = \int_0^1 1 \, ds = t$, so $M_t^2 - t$ is also a local martingale. It follows from Lévy’s theorem that $M$ is also an $(\mathcal{F}_t)$ Brownian motion.

**Example 2.** As in the previous example, let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space, and let $B = (B_t)_{t \geq 0}$ be an $(\mathcal{F}_t)$ Brownian motion. Now let $H = (H_t)_{t \geq 0}$ be an arbitrary element of $L^2_{\text{loc}}$. As before we define the local martingale

$$M_t := \int_0^t H_s dB_s, \quad t \geq 0,$$

which has quadratic variation process

$$\langle M \rangle_t = \int_0^t H_s^2 \, ds, \quad t \geq 0.$$
Observe that $\langle M \rangle_t$ is continuous and non-decreasing. Let us assume that, almost surely,

$$(5) \quad \lim_{t \to \infty} \langle M \rangle_t = \infty.$$ 

Define

$$T(s) := \inf \{ t : \langle M \rangle_t > s \}, \quad s \geq 0.$$ 

Then (5) implies that $T(s)$ is finite (a.s.) for each $s \geq 0$, and it is not hard to check that each $T(s)$ is a stopping time. We use these stopping times to “time change” $M$ into Brownian motion. Precisely, define

$$C_s := M_{T(s)}, \quad s \geq 0,$$

and

$$G_s := \mathcal{F}_{T(s)}, \quad s \geq 0.$$ 

Then the stochastic process $C = (C_s)_{s \geq 0}$ is adapted to the filtration $(G_s)_{s \geq 0}$. The optional stopping theorem implies that $C$ is a local martingale (with respect to $(G_s)$). Moreover, the quadratic variation interpretation of $\langle C \rangle$ and $\langle M \rangle$ implies that

$$\langle C \rangle_s = \langle M \rangle_{T(s)} = s, \quad \forall s \geq 0,$$

almost surely. That is, $C_s^2 - s$ is also a $(G_s)$ local martingale. Lévy’s theorem now tells us that $C = (C_s)$ is a $(G_s)$ Brownian motion. The moral: A continuous local martingale is just Brownian motion with its “clock” running too fast (or too slow).