Ex. 6.1.1. \( P_0(t) = e^{-t} \). For the rest I use the recursion (formula (6.5) on page 280 of the text)

\[ P_n(t) = \lambda_n - 1 \int_0^t e^{\lambda_n (t-s)} P_{n-1}(s) \, ds \]

for \( n = 1, 2, 3 \):

\[ P_1(t) = \int_0^t e^{-3(t-s)} P_0(s) \, ds = e^{-3t} \int_0^t e^{3s} \, ds = e^{-3t} (e^{2t} - 1)/2 = \frac{1}{2} e^{-t} - \frac{1}{2} e^{-3t}. \]

\[ P_2(t) = 3 \int_0^t e^{-2(t-s)} P_1(s) \, ds = \frac{3}{2} e^{-2t} \int_0^t (e^{s} - e^{-s}) \, ds = \frac{3}{2} e^{-2t} \left[ (e^t - 1) - (1 - e^{-t}) \right] = \frac{3}{2} (e^{-t} - 2e^{-2t} + e^{-3t}). \]

\[ P_3(t) = 2 \int_0^t e^{-5(t-s)} P_2(s) \, ds = 3e^{-5t} \int_0^t e^{5s} (e^{-s} - 2e^{-2s} + e^{-3s}) \, ds = 3e^{-5t} \int_0^t (e^{4s} - 2e^{3s} + e^{2s}) \, ds = 3e^{-5t} \left[ \frac{1}{4} (e^{4t} - 1) - \frac{2}{3} (e^{3t} - 1) + \frac{1}{2} (e^{2t} - 1) \right] = \frac{3}{4} e^{-t} - 2e^{-2t} + \frac{3}{2} e^{-3t} - \frac{1}{4} e^{-5t}. \]

Ex. 6.1.2. (a) \( W_3 = S_0 + S_1 + S_2 \), so

\[ E[W_3] = E[S_0] + E[S_1] + E[S_2] = \lambda_0^{-1} + \lambda_1^{-1} + \lambda_2^{-1} = 1 + \frac{1}{3} + \frac{1}{2} = \frac{11}{6}. \]

(b) Similarly, \( E[W_1] = 1 \) and \( E[W_2] = 4/3 \), so \( E[W_1 + W_2 + W_3] = 1 + 4/3 + 11/6 = 25/6. \)

(c) The variance of \( W_3 \) is the sum of the variances of \( S_0 \), \( S_1 \), and \( S_2 \). We know that the variance of an exponentially distributed random variable with parameter \( \lambda \) is \( 1/\lambda^2 \). Therefore,

\[ \text{Var}[W_3] = 1 + \frac{1}{9} + \frac{1}{4} = \frac{49}{36}. \]

Ex. 6.1.5. According to formula (6.10) (page 282 of the text), if \( X(0) = 1 \) then \( X(t) \) has the geometric distribution with parameter \( p = e^{-\beta t} \). From known formulas for the mean and variance of a geometric random variable we deduce that

\[ E[X(t)] = e^{\beta t} \]
and
\[ \text{Var}[X(t)] = e^{2\beta t}(1 - e^{-\beta t}). \]

(Cf. the formulae for the moments of \( Z' \) on page 21 of the text.)

**Pr. 6.1.3.** If, at time \( t \), there are \( X(t) \) infected individuals, then at that time there are \( N - X(t) \) susceptible individuals in the population. Since the individual infection rate (for each infected/susceptible pair) is \( \alpha \), the total infection rate, when \( X(t) = k \), is \( \lambda_k = \alpha k(N - k) \). That is, \( \lambda_k = \alpha k(N - k) \) for \( k = 0, 1, 2, \ldots, N \).

**Pr. 6.1.8.** Evidently,
\[ P_0(t + h) = P_0(t)(1 - \beta h) + P_1(t)\alpha h + o(h), \quad h \to 0+. \]

Consequently,
\[
\frac{P_0(t + h) - P_0(t)}{h} = -\beta P_0(t) + \alpha P_1(t) + \frac{o(h)}{h},
\]
whence
\[ P_0'(t) = -\beta P_0(t) + \alpha P_1(t). \]

Similarly,
\[ P_1'(t) = -\alpha P_1(t) + \beta P_0(t). \]

Because \( P_1(t) = 1 - P_0(t) \), equation (6.1.8.1) can be rewritten as
\[ P_0'(t) = - (\beta + \alpha) P_0(t) + \alpha. \]

Thus,
\[ P_0'(t) + (\beta + \alpha) P_0(t) = \alpha, \]
so
\[
\frac{d}{dt} \left( e^{(\alpha + \beta)t} P_0(t) \right) = \alpha e^{(\alpha + \beta)t}.
\]

Integrating we find that
\[ e^{(\alpha + \beta)t} P_0(t) - 1 = \frac{\alpha}{\alpha + \beta} \left( e^{(\alpha + \beta)t} - 1 \right), \]
because \( P_0(0) = 1 \). It follows that
\[ P_0(t) = \frac{\alpha}{\alpha + \beta} + \frac{\beta}{\alpha + \beta} e^{-(\alpha + \beta)t}. \]

Because \( P_1(t) = 1 - P_0(t) \), we also have
\[ P_1(t) = \frac{\beta}{\alpha + \beta} - \frac{\beta}{\alpha + \beta} e^{-(\alpha + \beta)t}. \]

**Pr. 6.1.9.** [This problem, as stated in the text, is much harder than the authors may have intended. To make the problem do-able, as discussed in class, we assume that \( N(t) \) is a pure birth process \( N(t) \) with birth rates \( \lambda_k = \alpha \) if \( k \) is odd and \( \lambda_k = \beta \) if \( k \) is even.]

Let \( P_k(t) = P[N(t) = k] \) for \( k = 0, 1, 2, \ldots \) and \( M(t) = E[N(t)] \). Let us also write \( \eta(t) \) for \( P[N(t) \text{ is even}] \). (This was called \( P_0'(t) \) in Problem 6.1.8.) We know that
\[ P_{2k}'(t) = -\beta P_{2k}(t) + \alpha P_{2k-1}(t), \quad k = 1, 2, \ldots, \]
and

\[ P'_{2k+1} = -\alpha P_{2k+1}(t) + \beta P_{2k}(t), \quad k = 0, 1, 2, \ldots. \]

Therefore,

\[
M'(t) = \sum_{k=1}^{\infty} 2kP'_k(t) + \sum_{k=0}^{\infty} (2k + 1)P'_{2k+1}(t)
\]

\[ = -\beta \sum_{k=1}^{\infty} 2kP_k(t) + \alpha \sum_{k=1}^{\infty} 2kP_{2k-1}(t) - \alpha \sum_{k=0}^{\infty} (2k + 1)P_{2k+1}(t) + \beta \sum_{k=0}^{\infty} (2k + 1)P_{2k}(t) \]

\[ = \beta \eta(t) + \alpha (1 - \eta(t)) \]

\[ = \alpha + (\beta - \alpha)\eta(t). \]

Using the formula for \( \eta(t) \) from problem 6.1.8, we see that

\[
M'(t) = \alpha + (\beta - \alpha) \left( \frac{\alpha}{\alpha + \beta} + \frac{\beta}{\alpha + \beta} e^{-(\alpha + \beta)t} \right).
\]

Since \( M(0) = 0 \), we must have

\[
M(t) = \frac{2\alpha \beta}{\alpha + \beta} t + \frac{\beta(\beta - \alpha)}{(\alpha + \beta)^2} (1 - e^{-(\alpha + \beta)t}).
\]

**Ex. 6.2.1.** Using the formulas on page 287 of the text (and a little patience), we find that

\[
P_3(t) = e^{-5t}
\]

\[
P_2(t) = \frac{5}{3} \left[ e^{-2t} - e^{-5t} \right]
\]

\[
P_1(t) = \frac{5}{3} \left[ 2e^{-2t} - 3e^{-3t} + e^{-5t} \right],
\]

and of course \( P_0(t) = 1 - P_1(t) - P_2(t) - P_3(t) \), so that

\[
P_0(t) = 1 - 5e^{-2t} + 5e^{-3t} - e^{-5t}.
\]

As a check, note that \( P_3(0) = 1, P_2(0) = P_1(0) = P_0(0) = 0 \), and

\[
P'_3(t) = -5P_3(t)
\]

\[
P'_2(t) = -2P_2(t) + 5P_3(t)
\]

\[
P'_1(t) = -3P_1(t) + 2P_2(t), \quad \text{and}
\]

\[
P'_0(t) = 3P_1(t),
\]

as expected.

Alternatively, you can use the formula \( P_3(t) = e^{-\mu_3 t} \), and the recursion discussed in class:

\[
P_k(t) = \mu_{k+1} \int_0^t e^{-\mu_3 (t-s)} P_{k+1}(s) \, ds, \quad k = 2, 1, 0,
\]

to compute (in succession) \( P_2(t), P_1(t), \) and \( P_0(t) \).

**Ex. 6.2.2.** (a) \( W_3 = S_3 + S_2 + S_1 \), so

\[
\mathbb{E}[W_3] = \mathbb{E}[S_3] + \mathbb{E}[S_2] + \mathbb{E}[S_1] = \frac{1}{5} + \frac{1}{2} + \frac{1}{3} = \frac{31}{30}.
\]
(c) $\text{Var}[W_3] = \text{Var}[S_3] + \text{Var}[S_2] + \text{Var}[S_1] = 1/25 + 1/4 + 1/9 = 361/900 = .4011\ldots$

**Pr. 6.2.2.** Since the death rates are all the same (namely $\theta$), the sojourn times in the various states all have the same exponential distribution, and their sums have gamma distributions. More precisely, for $k = 1, 2, \ldots N$, the random variable $W_k = S_N + S_{N-1} + \cdots + S_{N-k+1}$ has the gamma distribution with parameters $\theta$ and $k$; that is, the density function of $W_k$ is

$$f_{W_k}(t) = \frac{\theta^k t^{k-1} e^{-\theta t}}{(k-1)!}, \quad t > 0.$$ 

Therefore

$$P[X(t) = n] = P[W_{N-n} \leq t < W_{N-n+1}] = P[W_{N-n} \leq t] - P[W_{N-n+1} \leq t], \quad n = 1, 2, \ldots, N.$$ 

But we know from studying Poisson processes that

$$P[W_k \leq t] = \sum_{j=k}^{\infty} e^{-\theta t} \frac{(\theta t)^j}{j!}.$$ 

Combining this with the last-displayed equation we find that

$$P[X(t) = n] = e^{-\theta t} \frac{(\theta t)^{N-n}}{(N-n)!}, \quad n = 1, 2, \ldots, N.$$ 

Similarly,

$$P[X(t) = 0] = P[W_N \leq t] = \sum_{j=N}^{\infty} e^{-\theta t} \frac{(\theta t)^j}{j!}.$$ 

**Pr. 6.2.3.** A glance at the picture on page 287 of the text should be enough to convince you that the area under the trajectory of the pure death process is

$$\sum_{k=1}^{N} k \cdot S_k.$$ 

Consequently, the desired expectation is

$$\sum_{k=1}^{N} \frac{k}{\mu_k}.$$