Ex. 8.1.6.  (a) Because $B(u)$ and $B(u + v)$ have a bivariate normal distribution, the random variable $B(u) + B(u + v)$ is normally distributed with mean

$$E[B(u) + B(u + v)] = E[B(u)] + E[B(u + v)] = 0 + 0 = 0$$

and variance

$$\text{Var}(B(u) + B(u + v)) = \text{Var}(B(u)) + \text{Var}(B(u + v)) + 2\text{Cov}(B(u), B(u + v))$$

$$= u + (u + v) + 2u = 4u + v.$$ 

(b) Let $Y$ denote $B(u) + B(u + v) + B(u + v + w)$. As in part (a), the random variable $Y$ is normally distributed with mean 0. The conditional distribution of $Y$ given $B(u)$ and $B(u + v)$ is normal with mean $B(u) + 2B(u + v)$ and variance $w$. The variance of $Y$ is therefore the mean of this (conditional) variance (namely, $w$) plus the variance of the conditional mean $E[Y | B(u), B(u + v)] = B(u) + 2B(u + v)$ (which is equal to $u + 4(u + v) + 4u = 9u + 4v$). Thus,

$$\text{Var}(Y) = 9u + 4v + w.$$ 

Pr. 8.1.3. Because $B(t)/\sqrt{t}$ has the standard normal distribution,

$$P\left[\frac{|B(t)|}{\sqrt{t}} > \epsilon\right] = P\left[\frac{|B(t)|}{\sqrt{t}} > \epsilon\sqrt{t}\right] = 2[1 - \Phi(\epsilon\sqrt{t})].$$

As $t \to \infty$, this probability tends to zero, because $\Phi(z) \to 1$ as $z \to +\infty$. As $t \to 0+$, this probability tends to 1, because $\Phi(z) \to 1/2$ as $z \to 0$.

Pr. 8.1.7. Because $B(n)$ is a Markov chain, if $f$ is any function then

$$E[f(B(n + 1)) | B(0), B(1), \ldots, B(n)] = E[f(B(n + 1)) | B(n)].$$

Also, we know that the conditional distribution of $B(n + 1)$, given $B(n)$, is normal with mean $B(n)$ and variance 1. Therefore,

(a) $$E[B(n + 1) | B(n)] = B(n),$$

which shows that $B(n)$ is a martingale. In the same way,

$$E[B(n + 1)^2 | B(n)] = 1 + B(n)^2.$$ 

Because this latter conditional expectation depends only on $B(n)^2$, it follows (at least intuitively) that

$$E[B(n + 1)^2 | B(0)^2, B(1)^2, \ldots, B(n)^2] = E[B(n + 1)^2 | B(n)^2]$$

$$= 1 + B(n)^2.$$
Consequently,

\[(b) \ E[B(n + 1)^2 - (n + 1)|B(0)^2, B(1)^2, \ldots, B(n)^2] = 1 + B(n)^2 - n - 1 = B(n)^2 - n,\]

which shows that \(B(n)^2 - n\) is a martingale.

**Ex. 8.2.1.**

(a) \(\Pr[M(4) \leq 2] = 2\Phi(2/\sqrt{4}) - 1 = 2\Phi(1) - 1 = 2(0.8413) - 1 = 0.6826.\)

(b) \(\Pr[M(9) > c] = 2[1 - \Phi(c/3)],\) so for this to equal .1 we should have \(\Phi(c/3) = .95,\) or \(c/3 = 1.645,\) or \(c = 4.935.\)

**Ex. 8.2.3.** This is the chance that, for a Brownian motion started at 0, the first time to hit \(-3.29\) is larger than 4; namely,

\[\Pr[M(4) < 3.29] = 2\Phi(3.29/\sqrt{4}) - 1 = 2\Phi(1.645) - 1 = 2(0.95) - 1 = 0.9.\]

**Pr. 8.2.1.** Let \(A\) (respectively \(B\)) be the event that a standard Brownian motion is never 0 in the interval \((t, t + a]\) (respectively \((t, t + b].\)) We require

\[
\Pr[B|A] = \frac{\Pr[AB]}{\Pr[A]} = \frac{\Pr[B]}{\Pr[A]} = \frac{1 - \vartheta(t, t + b)}{1 - \vartheta(t, t + a)} = \frac{1 - (2/\pi) \arccos \sqrt{t/(t + b)}}{1 - (2/\pi) \arccos \sqrt{t/(t + a)}} = \frac{(2/\pi) \arcsin \sqrt{t/(t + b)}}{(2/\pi) \arcsin \sqrt{t/(t + a)}} = \frac{\arcsin \sqrt{t/(t + b)}}{\arcsin \sqrt{t/(t + a)}}.
\]

**Pr. 8.2.2.** The required probability is the limit as \(t \to 0^+\) of the conditional probability found in Problem 8.2.1. For this we need only recall that

\[\lim_{x \to 0} \frac{\arcsin(x)}{x} = 1.\]

Therefore the desired conditional probability is

\[
\lim_{t \to 0} \frac{\arcsin \sqrt{t/(t + b)}}{\arcsin \sqrt{t/(t + a)}} = \lim_{t \to 0} \sqrt{t/(t + b)} = \lim_{t \to 0} \sqrt{(t + a)/(t + b)} = \sqrt{a/b}.
\]