Ex. 8.2.6. Evidently, \( \{ \tau_1 < t \} \) if and only if the Brownian motion has a zero in the time interval \((b, t)\). According to formula (8.25) (page 408), the probability of this event is \( \vartheta(b, t) = \frac{2}{\pi} \arccos \sqrt{b/t} \).

Pr. 8.2.4. [Note the typo: \( m \) in the third line of the problem statement should be \( z \).] For \( 0 < x < z \), the reflection principle tells us that the event \( \{ M(t) \geq z, B(t) \leq x \} \) has the same probability as the event \( \{ M(t) \geq z, 2z - B(t) \leq x \} = \{ M(t) \geq z, B(t) \geq 2z - x \} \). Also, in the latter event, the condition \( M(t) \geq z \) is redundant, being implied by the other condition. Therefore,

\[
P[M(t) \geq z, B(t) \leq x] = P[B(t) \geq 2z - x] = 1 - \Phi((2z - x)/\sqrt{t}).
\]

Differentiating the right side with respect to \( x \) we obtain

\[
\frac{1}{\sqrt{t}} \phi \left( \frac{2z - x}{\sqrt{t}} \right),
\]

and differentiating \( this \) with respect to \( z \) we obtain

\[
(1) \quad \frac{2}{t} \phi' \left( \frac{2z - x}{\sqrt{t}} \right).
\]

But for the standard normal density \( \phi \), we have the identity \( \phi'(y) = -y\phi(y) \), so \( 1 \) can be written as

\[
-\frac{2}{t} \cdot \frac{2z - x}{\sqrt{t}} \phi \left( \frac{2z - x}{\sqrt{t}} \right).
\]

The desired joint density function of \( M(t) \) and \( B(t) \) is therefore

\[
\frac{2}{t} \cdot \frac{2z - x}{\sqrt{t}} \phi \left( \frac{2z - x}{\sqrt{t}} \right).
\]

Pr. 8.2.6. The marginal density of \( Y(t) \) is gotten by integrating out \( z \) (from 0 to \(+\infty\)) in the given joint density function:

\[
f_{Y(t)}(y) = \int_0^\infty \frac{z + y}{t} \cdot \frac{2}{\sqrt{t}} \phi \left( \frac{z + y}{\sqrt{t}} \right) dz
\]

\[
= \frac{2}{\sqrt{t}} \int_{y/\sqrt{t}}^\infty u\phi(u) du.
\]

But an antiderivative of

\[
u\phi(u) = \frac{u}{\sqrt{2\pi}}e^{-u^2/2}
\]

is

\[
-\frac{1}{\sqrt{2\pi}}e^{-u^2/2};
\]

1
so
\[ f_Y(y) = -\frac{2}{\sqrt{2\pi t}} e^{-u^2/2} \left| \frac{y}{\sqrt{t}} \right| = \frac{2}{\sqrt{2\pi t}} e^{-y^2/2t}. \]

Of course, this is twice the normal density with mean 0 and variance \( t \) (restricted to the positive axis), which is the density function of \(|B(t)|\).

**Ex. 8.3.2.** We are asked to compute \( P[A(25) = 0|A(0) = 5] \) and \( P[A(25) \geq 10|A(0) = 5] \). We use formula (8.32) on page 414:

\[
P[A(25) = 0|A(0) = 5] = 1 - \Phi(\frac{5}{\sqrt{25}}) + \Phi(\frac{-5}{\sqrt{25}})
= 1 - \Phi(1) + \Phi(-1) = 2[1 - \Phi(1)] = .3174,
\]

and
\[
P[A(25) > 10|A(0) = 5] = \Phi(\frac{15}{\sqrt{25}}) - \Phi(\frac{5}{\sqrt{25}}) = \Phi(3) - \Phi(1)
= .9987 - .8413 = .1575.
\]

**Ex. 8.3.3.**
\[
P[R(25) > 10|R(0) = 5] = P[|B(25)| > 10|B(0) = 5]
= 1 - P[|B(25)| \leq 10|B(0) = 5]
= 1 - \int_{-10}^{10} \phi_25(y - 5) \, dy
= 1 - \int_{-10}^{10} \phi((y - 5)/5) \, dy\frac{dy}{5}
= 1 - \int_{-3}^{1} \phi(z) \, dz
= \Phi(-3) + 1 - \Phi(1)
= 2 - \Phi(3) - \Phi(1) = 2 - .9987 - .8413 = .16.
\]

**Pr. 8.3.3.** Both \( B(u) - uB(1) \) and \( B(1) \) have mean 0, and their covariance is

\[
E[(B(u) - uB(1))B(1)] = E[B(u)B(1)] - uE[B(1)B(1)] = \min(u, 1) - u = u - u = 0,
\]

so \( B(u) - uB(1) \) and \( B(1) \) are independent.

(a) We have \( B(t) = B^o(t) + tB(1) \), and by the preceding discussion, the process \( B^o \) is independent of the random variable \( B(1) \). Consequently, the conditional distribution of \( B \) given that \( B(1) = 0 \) (namely, the distribution of a Brownian bridge) is the same as the unconditional distribution of \( B^o \). In other words, \( B^o \) has the Brownian bridge distribution.

(b) In view of part (a) the covariance function of the Brownian bridge is (for \( 0 \leq s \leq t \leq 1 \))

\[
E[B^o(s)B^o(t)] = E[(B(s) - sB(1)) \cdot (B(t) - tB(1))]
= E[B(s)B(t) - sB(t)B(1) - tB(s)B(1) + stB(1)^2]
= s - st - ts + st = s - st = s(1 - t).
\]