1. Problem 10 (page 549)

2. Problem 15 (page 550) (Restated here with some clarifications.) The Riesz decomposition, Theorem 5.3 (p. 491), states that any supermartingale \(\{(X_n, \mathcal{F}_n), n \geq 0\}\) satisfying \(\inf_n \mathbb{E}[X_n] > -\infty\) can be uniquely decomposed into a martingale, \(\{(M_n, \mathcal{F}_n), n \geq 0\}\), and a potential, \(\{(Z_n, \mathcal{F}_n), n \geq 0\}\)

\[ X_n = M_n + Z_n. \]

(A potential is a non-negative supermartingale \(\{(Z_n, \mathcal{F}_n), n \geq 0\}\) with the additional property that \(\lim_{n \to \infty} \mathbb{E}[Z_n] = 0\).)

(a) Explain why there exists a martingale \(\{(V_n, \mathcal{F}_n), n \geq 0\}\) and a predictable increasing process \(\{(A_n, \mathcal{F}_n), n \geq 0\}\) with \(A_0 = 0\) such that

\[ X_n = V_n - A_n, \quad n \geq 0. \]

(b) Check that \(\mathbb{E}[A_n] \leq \mathbb{E}[V_0] - \inf_n \mathbb{E}[X_n]\), for all \(n \geq 0\).

(c) Prove that \(A_\infty := \lim_n A_n\) exists almost surely, and that \(\mathbb{E}[A_\infty] < \infty\).

(d) Set

\[ M_n := V_n - \mathbb{E}[A_\infty | \mathcal{F}_n] \quad \text{and} \quad Z_n := \mathbb{E}[A_\infty | \mathcal{F}_n] - A_n, \quad n \geq 0, \]

and prove that this is a decomposition of the desired kind.

(e) Prove that the decomposition is unique.

3. Let \(\{\mathcal{F}_n : n \geq 0\}\) be a filtration on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), and let \(\{(X_n, \mathcal{F}_n) : n \geq 0\}\) and \(\{(Y_n, \mathcal{F}_n) : n \geq 0\}\) be submartingales. Show that \(\{Z_n\}\) defined by

\[ Z_n := X_n \vee Y_n = \max(X_n, Y_n), \quad n = 0, 1, 2, \ldots, \]

is also a submartingale.

4. Let \(\{X_n : n \geq 0\}\) be a martingale such that \(\mathbb{P}[X_n = 0] = 0\) and \(X_{n+1}/X_n \in L^1\) for all \(n \geq 0\). Define \(Y_n := X_{n+1}/X_n\).

(a) Show that \(\mathbb{E}[Y_n] = 1\) for all \(n \geq 0\).

(b) Show that the random variables \(Y_n\) and \(Y_{n-1}\) are uncorrelated for each \(n \geq 1\). (That is, \(\text{Cov}[Y_n, Y_{n-1}] = 0, n \geq 1\).)

5. Let \(\{X_n : n \geq 0\}\) be a non-negative supermartingale. Show that the state 0 is an “absorbing point” for \(\{X_n\}\) in the sense that

\[ \mathbb{P}[X_m = 0, \sup\{X_n : n > m\} > 0] = 0, \quad \forall m \geq 0. \]