1. Problem 17 (page 551)

2. Problem 22 (page 551)

3. Let \((X_n)\) be a martingale and suppose there is a constant \(C \in \mathbb{R}\) such that \(X_n \leq C\) almost surely for each \(n\). Prove that \(\sup_n E[|X_n|] < \infty\); i.e., that \((X_n)\) is \(L^1\)-bounded.

4. Let \(Y_1, Y_2, \ldots\) be independent random variables with \(E[Y_k] = 0\) and \(E[Y_k^2] < \infty\) for all \(k\). Use martingales to prove Kolmogorov’s convergence criterion: If \(\sum_{k=1}^{\infty} E[Y_k^2] < \infty\) then \(\sum_{k=1}^{\infty} Y_k\) converges almost surely.

5. Let \(Y_0, Y_1, Y_2, \ldots\) be non-negative and integrable random variables. Define \(\mathcal{F}_n := \sigma\{Y_0, Y_1, \ldots, Y_n\}\) for \(n \geq 0\), and suppose that

\[
E[Y_{n+1} | \mathcal{F}_n] \leq b_n + Y_n, \quad n = 0, 1, 2, \ldots,
\]

where \((b_n)_{n=0}^{\infty}\) is a sequence of non-negative constants with \(\sum_{n=0}^{\infty} b_n < \infty\). Show that

(a) \(Y_\infty := \lim_{n \to \infty} Y_n\) exists almost surely, and

(b) \(Y_\infty\) is integrable.

[Hint: Begin by showing that \(Y_n + \sum_{k=n}^{\infty} b_k, n \geq 0,\) is a non-negative supermartingale.]