Quiz 1.

1. Find the quadratic approximation to \( f(x) = \frac{2}{1 + x} \), at \( a = 3 \).

\[
\begin{align*}
    f'(x) &= -\frac{2}{(1 + x)^2}, \\
    f''(x) &= \frac{4}{(1 + x)^3}.
\end{align*}
\]

So, \( f(3) = \frac{1}{2}, \quad f'(3) = -\frac{1}{8}, \quad f''(3) = \frac{1}{16}, \) and

\[
f(x) \approx f(3) + f'(3)(x - 3) + \frac{1}{2}f''(3)(x - 3)^2 = \frac{1}{2} - \frac{1}{8}(x - 3) + \frac{1}{32}(x - 3)^2.
\]

2. Find the Maclaurin expansion for \( f(x) = x \sin 2x \). We know that

\[
\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.
\]

Setting \( z = 2x \), we have

\[
x \sin x = x \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{(2n+1)!} x^{2n+2}.
\]

3. (a) By L’Hospital’s rule, as \( x \to 0 \),

\[
\frac{\sin(x^2)}{2x^2} \sim \frac{2x \cos(x^2)}{4x} = \frac{\cos(x^2)}{2} \to \frac{1}{2}.
\]

(b) By the same rule, as \( x \to \infty \),

\[
x^2 \left( e^{1/x^2} - 1 \right) = \frac{e^{1/x^2} - 1}{1/x^2} \sim \frac{(-2/x^3)e^{1/x^2}}{-2/x^3} = e^{1/x^2} \to 1.
\]

(c) Let \( y = (3 + x)^{1/\sqrt{x}} \). Then, as \( x \to \infty \),

\[
\ln y = \frac{\ln(3 + x)}{\sqrt{x}} \sim \frac{1/(3 + x)}{1/(2\sqrt{x})} = \frac{2\sqrt{x}}{3 + x} \sim \frac{1/\sqrt{x}}{1} = \frac{1}{\sqrt{x}} \to 0.
\]

Thus, \( \ln y \to 0 \), and consequently, \( y \to 1 \).
Quiz 2.

1. \( \int_3^3 e^{2x} \, dx = 0 \), since this integral is the area under the graph of the function \( e^{2x} \), and over the segment \([3, 3]\). This segment is just a point, and its length equals zero.

2. One of antiderivatives of the function

\[
 f(x) = 3e^{2x} + \frac{1}{5}\sin 10x + x^11
\]

is the function

\[
 F(x) = \frac{3}{2}e^{2x} - \frac{1}{50}\cos 10x + \frac{1}{12}x^{12}.
\]

All antiderivatives are described by the formula \( F(x) + C \).

3. (a) \( \int_0^1 \frac{x}{1 + 2x^2} \, dx \); Let \( u = 1 + 2x^2 \). Then \( du = 4x \, dx \), and

\[
 \int_0^1 \frac{x}{1 + 2x^2} \, dx = \frac{1}{4} \int_1^3 \frac{1}{u} \, du = \frac{1}{4} \ln u\bigg|_1^3 = \frac{\ln 3}{4}.
\]

(b) \( \int_0^{\ln^{3/2}} \frac{e^x}{1 + e^{2x}} \, dx \). Let \( u = e^x \). Then \( du = e^x \, dx \), and

\[
 \int_0^{\ln^{3/2}} \frac{e^x}{1 + e^{2x}} \, dx = \int_1^{\sqrt{3}} \frac{1}{u^2} \, du = \arctan \frac{\sqrt{3}}{1} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.
\]

4. Find the volume of revolution of the region bounded by \( y = e^{2x} \), \( y = 0 \), \( x = 0 \), \( x = 1 \) about the \( x \)-axis. The volume equals

\[
 \pi \int_0^1 e^{4x} \, dx = \pi \left. \frac{e^{4x}}{4} \right|_0^1 = \pi \left( e - 1 \right).
\]

MIDTERM.

1. \( \lim_{x \to 0} \frac{1 - \cos(x^2)}{2x^4} = \lim_{x \to 0} \frac{2x \sin(x^2)}{8x^3} = \lim_{x \to 0} \frac{\sin(x^2)}{4x^2} = \lim_{x \to 0} \frac{2x \cos(x^2)}{8x} = \lim_{x \to 0} \frac{\cos(x^2)}{4} = \frac{1}{4} \).

2. (a) Integrating by parts, we have

\[
 \int_1^e \ln^2 x \, dx = \left[ \ln^2 x \right]_1^e - \int_1^e 2\ln x \, dx = e - 0 - 2 \left[ \ln x \right]_1^e = e - 2 \left( \ln e - \int_1^e \frac{1}{x} \, dx \right) = e - 2 \left( e - \int_1^e dx \right) = e - 2.
\]
(b) Setting \( u = \cos x \), we have \( du = -\sin x \, dx \), and
\[
\int \cos^2 x \sin^3 x \, dx = \int \cos^2 x \sin^2 x (\sin x) \, dx = \int \cos^2 x (1 - \cos^2 x)(\sin x) \, dx \\
= -\int u^2 (1 - u^2) \, du = -\frac{u^3}{3} + \frac{u^5}{5} + C = -\frac{(\cos x)^3}{3} + \frac{(\cos x)^5}{5} + C.
\]

(c) Setting \( z = \tan x \), we have \( dx = dt/\cos^2 t \), \( 1 + x^2 = 1/\cos^2 t \), and
\[
\int_0^1 \frac{dx}{(1 + x^2)^{3/2}} = \int_0^{\pi/4} \frac{dt}{\cos^2 t (1/\cos^2 t)^{3/2}} = \int_0^{\pi/4} \frac{\cos^3 t \, dt}{\cos^2 t} = \int_0^{\pi/4} \cos t \, dt = \frac{\sqrt{2}}{2}.
\]

(e) We write
\[
\frac{x^2}{x^3 + x^2 + x + 1} = \frac{x^2}{(x + 1)(x^2 + 1)} = \frac{a}{x + 1} + \frac{bx + c}{x^2 + 1}
\]
\[
= \frac{ax^2 + a + bx^2 + cx + bx + c}{(x + 1)(x^2 + 1)} = \frac{(a + b)x^2 + (b + c)x + a + c}{(x + 1)(x^2 + 1)}.
\]
Hence, \( a + b = 1 \), \( b + c = 0 \), \( a + c = 0 \). From this, \( a = b = 1/2 \), \( c = -1/2 \). Thus,
\[
\int \frac{x^2}{x^3 + x^2 + x + 1} \, dx = \frac{1}{2} \left\{ \int \left( \frac{1}{x + 1} + \frac{x - 1}{x^2 + 1} \right) \, dx \right\}
\]
\[
= \frac{1}{2} \left( \int \frac{1}{x + 1} \, dx + \int \frac{x}{x^2 + 1} \, dx - \int \frac{1}{x^2 + 1} \, dx \right)
\]
\[
= \frac{1}{2} \left( \ln |x + 1| + \frac{1}{2} \ln(x^2 + 1) - \arctan x \right).
\]

3. Do the integrals below converge?

(a) If \( x \geq 1 \), then \( x^{10} \geq x \), \( x^{10} - x \geq 0 \), and
\[
\int_1^\infty \frac{dx}{2x^{10} - x + 3} = \int_1^\infty \frac{dx}{x^{10} + (x^{10} - x) + 3} \leq \int_1^\infty \frac{dx}{x^{10}}.
\]
The last integral converges. (See the theory.)

(b) Set \( u = 1 - x \). Then
\[
\int_0^1 \frac{dx}{\sqrt{1-x}} = -\int_1^0 \frac{du}{\sqrt{u}} = \int_0^1 \frac{du}{\sqrt{u}}.
\]
The last integral converges. (See the theory.)

4. Setting \( u = \ln x \), we have \( du = dx/x \), and
\[
\int_{\ln 2}^\infty \frac{dx}{x \ln^4 x} = \int_{\ln 2}^\infty \frac{du}{u^4} = u^{-3} \bigg|_{\ln 2}^\infty = -\frac{1}{3u^3} \bigg|_{\ln 2}^\infty = -0 + \frac{1}{3(\ln 2)^3} = \frac{1}{3(\ln 2)^3}.
\]