Finding Gaussian Binomial Identities with Integer Partitions

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1 Introduction

The Gaussian binomial coefficient

$$\begin{bmatrix} n \\ d \end{bmatrix}_q = \frac{\prod_{i=1}^{d}(1-q^{n+1-i})}{\prod_{i=1}^{d}(1-q^i)}$$

can be decomposed into sums of q-numbers in the form $[k] = \frac{1-q^k}{1-q}$ multiplied by powers of q, so the purpose of the paper is to find a systematic way to identify the decompositions of $\begin{bmatrix} n \\ d \end{bmatrix}_q$ and how they relate for a fixed d, particularly for $d \geq 3$. The main application of this decomposition is in representation theory – specifically the representation from the Lie algebra $sl_2(C)$ to $gl(\wedge^d(V_n))$, where V_n is the set of homogeneous polynomials of degree n. This representation, often denoted as $\wedge^d(V_n)$, can be decomposed into $\oplus V_\alpha$, where for each α, the character of V_α can be written as $[\alpha+1]$. Then the character of the representation $\wedge^d(V_n)$ can be written as $\begin{bmatrix} n+1 \\ d \end{bmatrix}_q$ and can also be written as the sums of characters of the representation of each V_α, so the Gaussian binomial coefficient identities can indicate the decomposition of $\wedge^d(V_n)$.

The decomposition of $\wedge^d(V_n)$ can be calculated by the mathematics software LiE or by manually considering the eigenvalues of V_n. In particular, if the eigenvalues of V are $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, then the eigenvalues of $\wedge^r V$ are in the multiset $\{\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_r}\}$ where $1 \leq i_1 < i_2 < \cdots < i_r \leq n$. The following two functions are then used as an aid to finding decompositions.

- Let $S(d, n, a)$ be the number of ways to choose d different elements from $\{-n, -n+2, \cdots, n-2, n\}$ to sum to the number a.
- Let $C(d, n, a) = S(d, n, a) - S(d, n, a+2)$.

If $a \geq 0$, then the value of $C(d, n, a)$ is the multiplicity of V_a in the decomposition of $\wedge^d(V_n)$. Additionally, by using a bijection from the set $\{-n, -n+2, \cdots, n-2, n\}$ to the set $\{1, 2, \cdots, n+1\}$, the values of $S(d, n, a)$ and $C(d, n, a)$ can be computed by using integer partitions of fixed part number and maximum part size. Along with the recurrence relation

$$C(d, n+1, a+d) = C(d, n, a) + C(d-1, n, a+n+2)$$

with the base case $C(d, n, dn - d(d-1)) = 1$, I later show that the generating series $F_{n,d}(x) = \sum_{k \geq 0} C(d, n, dn - d(d-1) - 2k)x^k$ is equal to

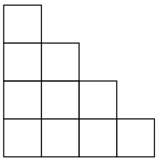
$$\left(\sum_{j=1}^{d} \frac{(-1)^{j+1} x^{(j-1)(n-d)+0.5j^2+0.5j-1}}{\prod_{i=1}^{(j-1)}(1-x^i)\prod_{i=2}^{d-(j-1)}(1-x^i)}\right) + \frac{(-1)^d x^{d(n-d)+0.5d(d+3)}}{\prod_{i=2}^{d}(1-x^i)}.$$

Expanding this generating series allows one to find the multiplicity of V_a for $a \geq 0$ in the decomposition of $\wedge^d(V_n)$. This generating series can also be used to identify recurrence relations and asymptotic behavior for constant d by taking first differences via $F_{n,d}(x) - x^{\frac{ds}{2}} F_{n-s,d}(x)$.

2 Background

2.1 Integer Partitions

First in discussion is integer partitions, which is an unordered set of positive integers that sum to another integer. For example, two examples of integer partition of 10 are $4+3+2+1$ and $2+2+2+2+1+1$. These integer partitions can be represented as a Young diagram, which lists congruent squares per row such that the edges of the leftmost squares all lie on a line and number of squares in each row is at least the number of squares in the row below. An example of a Young diagram for the partition $4+3+2+1$ is shown below.



Some integer partitions may impose the extra condition of having a maximum part size and having a maximum number of parts. The Young diagrams of such integer partitions can fit inside a rectangle. The following notation is used by MJ Kronenberg for some specific integer partitions. [3]

- $P(n,m)$ is the number of partitions of n into exactly m parts.
- $Q(n,m)$ is the number of partitions of n into exactly m distinct parts.
- $P^{\#}(n, p_{\min}, p_{\max})$ is the number of partitions of n such that each part is between $p_{\min}$ and $p_{\max}$ inclusive.
- $P(n,m,p)$ is the number of partitions of n into exactly m parts that are each at most p.
- $Q(n,m,p)$ is the number of partitions of n into exactly m distinct parts that are each at most p.

Additionally, the online OEIS database lists sequences for integer partitions and can be used as a reference for identifying sequence matches.

2.2 Generating Functions and Recurrence Relations

A formal power series $F(x)$ is a polynomial

$$F(x) = \sum_{i \geq 0} a_i x^i$$

that may have an infinite number of terms. This is also the generating function of the sequence $\{a_i\}$.

If for a formal power series $F(x)$ there is another formal power series $G(x)$ such that $F(x)G(x) = 1$, then $F(x)$ is invertible. Additionally, should $G(x)$ have a finite number of terms, the formal power series $F(x)$ can be written as $\frac{1}{G(x)}$, allowing for a more concise way to express formal power series. One example of an invertible formal power series is the formal power series $1 + x + x^2 + x^3 + \cdots$, which can be written as $\frac{1}{1-x}$.

In the context of integer partitions, the coefficient of x^m in the formal power series

$$\frac{1}{1-x^k} = 1 + x^k + (x^k)^2 + (x^k)^3 + \cdots$$

represents the number of integer partitions of m in which the only part size is k. This means that for $k_1 < k_2 < \cdots < k_j$, the coefficient of x^m in the formal power series

$$\frac{1}{\prod_{i=1}^{j}(1-x^{k_i})}$$

represents the number of integer partitions of m in which the part sizes are in the set $\{k_1, k_2, \cdots, k_j\}$.

2.3 Q-Numbers

Next in discussion is the q-numbers. The q-number is defined as

$$[k] := \begin{cases} \frac{1-q^k}{1-q} & \text{if } k > 0 \\ 1 & \text{if } k = 0 \end{cases}.$$

Additionally, $[k]!$ is defined as $\prod_{j=1}^{k}[j]$, and the Gaussian binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined as $\frac{[n]!}{[n-k]![k]!}$.

2.4 Direct Sums, Tensor Products, and Wedge Products

For vector spaces V and W, the direct sum $V \oplus W$ is a vector space such that each $u \in V \oplus W$ can be uniquely expressed as $u = v + w$ where $v \in V$ and $w \in W$. As such, u can be written in the form (v, w).

The tensor product $V \otimes W$ is a vector space such that the following holds for $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and a scalar a.

- $v_1 \otimes w + v_2 \otimes w = (v_1 + v_2) \otimes w$
- $v \otimes w_1 + v \otimes w_2 = v \otimes (w_1 + w_2)$
- $a(v \otimes w) = av \otimes w = v \otimes aw$

For vectors $v_1, v_2, \cdots, v_d \in V$, the wedge product is defined as

$$\frac{1}{d!} \sum_{\pi \in S(d)} \mathrm{sgn}(\pi) v_{\pi(1)} \otimes v_{\pi(2)} \otimes \cdots \otimes v_{\pi(d)}$$

where $\pi \in S(d)$ is a permutation of $\{1, 2, ..., d\}$. The vector space generated by all such elements is denoted as the exterior product $\wedge^d(V)$. Notably, for $v, w_1, w_2, \cdots, w_m \in V$, the anticommutativitiy property means that

$$v \wedge v \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_m = -v \wedge v \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_m$$

and so $v \wedge v \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_m = 0$. This means that if the basis of V is generated by n elements, then the dimension of $\wedge^d(V)$ is $\binom{n}{d}$.

2.5 Lie Algebra and Representation Theory

The most notable in discussion is Lie Algebra, which is a vector space with $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ where the following holds.

- $[x + y, z] = [x, z] + [y, z]$
- $[x, y + z] = [x, y] + [x, z]$
- $a[x, y] = [ax, y] = [x, ay]$
- $[x, y] = -[y, x]$
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

A representation of a Lie algebra $\mathfrak{g}$ on to vector space V is the map $\rho : \mathfrak{g} \to gl(V)$ where the following holds.

- $gl(V) = End(V)$, which is the set of linear transformation from V to itself
- $\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$

The representation $\rho : \mathfrak{g} \to gl(V)$ may sometimes be denoted as V. [4] If a representation has no non-trivial sub-representations, then that representation is irreducible.

The Lie algebra we focus on is

$$sl_2(C) = \{\begin{pmatrix} a & b \\ c & -a \end{pmatrix} | a, b, c \in C\},$$

which satisfies $[v, w] = vw - wv$ for $v, w \in sl_2(C)$ and has the basis $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. The following properties hold for $sl_2(C)$.

- $[H, X] = 2X$
- $[H, Y] = -2Y$
- $[X, Y] = H$

Now let V be a representation of $sl_2(C)$, and let U_α be the set of eigenvectors $v \in V$ where $H(v) = \alpha v$. Then $V = \oplus U_\alpha$ [4]. Additionally, $H(X(v)) = (\alpha+2)X(v)$, which leads to a set of eigenvalues of V. And for $u \in U_n$ where n is the largest eigenvalue, $\{u, Y(u), Y^2(u), \cdots\}$ would span the representation of V. This leads to unique $V^{(n)}$ with the set of eigenvalues of H being $\{n, n-2, \cdots, -n\}$. [1, 4]

The representation of $sl_2(C)$ we will focus on is $\rho : \mathfrak{g} \to gl(V_n)$ where V_n is the homogeneous polynomials of degree n with variables x and y and the following holds for the standard basis H, X, Y.

- $\rho(Y) = y\frac{\partial}{\partial x}$
- $\rho(X) = x\frac{\partial}{\partial y}$
- $\rho(H) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$

This representation is sometimes denoted as V_n, and the eigenvalues of the tranformation H is $\{n, n-2, \cdots, -n\}$. Additionally, the character of representation V_n is the polynomial

$$x^n + x^{n-1}y + x^{n-2}y^2 + \cdots + xy^{n-1} + y^n.$$

If $x = q$ and $y = 1$, the character would just be the q-coefficient $[n+1]$.

Additionally, for $x = q$ and $y = 1$, the character of representation $\wedge^d(V_n)$ is the q-binomial coefficient $\begin{bmatrix} n+1 \\ d \end{bmatrix}_q$. Since $\wedge^d(V_n)$ can be decomposed into $\oplus V_\alpha$, the character of representation $\wedge^d(V_n)$ can be decomposed uniquely into the sums of characters of V_a where $a \geq 0$.

The software LiE can be used to compute the decomposition of $\wedge^d(V_n)$ for some non-negative integer d and n with the following command.

```
> alt_tensor(d, X[n], A1)
```

Common differences can also be done with the following LiE command for some positive integer k.

```
> alt_tensor(d, X[n + k], A1) - alt_tensor(d, X[n], A1)
```

3 Findings

3.1 Basic Proofs

First I show some basic properties about Gaussian binomial coefficients using q-numbers, which would later be used for the Q-Binomial Theorem.

Theorem 3.1. $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$

Proof. By definition, $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[n-k]![k]!}$. Now $[n-k]![k]! = [k]![n-k]!$ by the Commutative Property, and $k = n-(n-k)$. Therefore, $\frac{[n]!}{[n-k]![k]!} = \frac{[n]!}{[n-(n-k)]![n-k]!}$, which by definition is $\begin{bmatrix} n \\ n-k \end{bmatrix}_q$. □

Theorem 3.2. $\begin{bmatrix} d \\ k \end{bmatrix}_q = \begin{bmatrix} d-1 \\ k \end{bmatrix}_q + \begin{bmatrix} d-1 \\ k-1 \end{bmatrix}_q q^{d-k}$

Proof. First we have

$$\begin{aligned} [d] &= 1 + q + \cdots + q^{d-k-1} + q^{d-k} + q^{d-k+1} + \cdots + q^{d-1} \\ &= (1 + q + \cdots + q^{d-k-1}) + q^{d-k}(1 + q + \cdots + q^{k-1}) \\ &= [d-k] + q^{d-k}[k]. \end{aligned}$$

Then multiplying both sides by $\frac{[d-1]!}{[k]![d-k]!}$ yields

$$\frac{[d]!}{[k]![d-k]!} = \frac{[d-1]!}{[k]![d-k-1]!} + q^{d-k}\frac{[d-1]!}{[k-1]![d-k]!},$$

which is indeed $\begin{bmatrix} d \\ k \end{bmatrix}_q = \begin{bmatrix} d-1 \\ k \end{bmatrix}_q + \begin{bmatrix} d-1 \\ k-1 \end{bmatrix}_q q^{d-k}$. □

Theorem 3.3 (Q-Binomial Theorem)**.**

$$\sum_{k=0}^{d} \begin{bmatrix} d \\ k \end{bmatrix}_q (-1)^k q^{\frac{k(k+1)}{2}} = \prod_{k=1}^{d} (1-q^k)$$

Proof. For the base case, if $d = 1$, both the left-hand side and the right-hand side equal $1 - q$.

For the inductive step, assume that

$$\sum_{k=0}^{d} \begin{bmatrix} d \\ k \end{bmatrix}_q (-1)^k q^{\frac{k(k+1)}{2}} = \prod_{k=1}^{d} (1-q^k).$$

Multiplying both sides by $-q^{d+1}$ and shifting the index results in

$$\sum_{k=0}^{d} \begin{bmatrix} d \\ k \end{bmatrix}_q (-1)^{k+1} q^{\frac{k(k+1)}{2}} q^{d+1} = -q^{d+1} \prod_{k=1}^{d} (1-q^k).$$

$$\sum_{k=1}^{d+1} \begin{bmatrix} d \\ k-1 \end{bmatrix}_q (-1)^k q^{\frac{(k-1)(k)}{2}} q^{d+1} = -q^{d+1} \prod_{k=1}^{d} (1-q^k).$$

Now consider $q^{\frac{k(k+1)}{2}} (\begin{bmatrix} d \\ k-1 \end{bmatrix}_q q^{d+1-k} + \begin{bmatrix} d \\ k \end{bmatrix}_q)$. From the q-binomial identities $\begin{bmatrix} d \\ k \end{bmatrix}_q = \begin{bmatrix} d \\ d-k \end{bmatrix}_q$ and $\begin{bmatrix} d \\ k \end{bmatrix}_q = \begin{bmatrix} d-1 \\ k \end{bmatrix}_q + \begin{bmatrix} d-1 \\ k-1 \end{bmatrix}_q q^{d-k}$, that expression is equivalent to $q^{\frac{k(k+1)}{2}} \begin{bmatrix} d+1 \\ k \end{bmatrix}_q$. Therefore, the value of

$$\sum_{k=0}^{d} \begin{bmatrix} d \\ k \end{bmatrix}_q (-1)^k q^{\frac{k(k+1)}{2}} + \sum_{k=1}^{d+1} \begin{bmatrix} d \\ k-1 \end{bmatrix}_q (-1)^k q^{\frac{(k-1)(k)}{2}} q^{d+1}$$

is equal to $\sum_{k=0}^{d+1} \begin{bmatrix} d \\ k \end{bmatrix}_q (-1)^k q^{k(k+1)}2$, which also equals

$$\left(\prod_{k=1}^{d} (1-q^k) \right) (1-q^{d+1}) = \prod_{k=1}^{d+1} (1-q^k).$$

$\square$

I also show the following theorem, which can provide a shorthand way for writing sums of some q-numbers.

Theorem 3.4. *For $a, b \geq 2$,*

$$[a][b] = \sum_{j=1}^{\min(a,b)} q^{j-1}[a+b-1-2(j-1)].$$

Proof. For the base case, $[1][1] = 1 = q^0[1+1-1-2(0)]$.

For the first part of the inductive step, assume $[a][a] = \sum_{j=1}^{a} \left(q^{j-1}[2a-1-2(j-1)] \right)$. Now $[a] = 1 + q + \cdots + q^{a-1}$, so

$$\begin{aligned}
[a+1][a+1] &= [a][a] + 2q^a(1+q+\cdots+q^{a-1}) + q^{2a} \\
&= \sum_{j=1}^{a} \left(q^{j-1}[2a-1-2(j-1)] \right) + 2q^a(1+q+\cdots+q^{a-1}) + q^{2a} \\
&= \sum_{j=1}^{a} \left(q^{j-1}[2a-1-2(j-1)]q^k \right) + 2q^a(1+q+\cdots+q^{a-1}) + q^{2a} \\
&= \sum_{j=1}^{a} \left(q^{j-1}[2a-1-2(j-1)] + q^{j-1}(q^{2a-1-2(j-1)} + q^{2a-2(j-1)}) \right) + q^a \\
&= \sum_{j=1}^{a+1} \left(q^{j-1}[2a+1-2(j-1)]q^k \right).
\end{aligned}$$

For the second part of the inductive step, assume $a \leq b$ and $[a][b] = \sum_{j=1}^{a} \left(q^{j-1}[a+b-1-2(j-1)]\right)$. Then

$$
\begin{aligned}
[a][b+1] &= [a][b] + [a]q^b \\
&= \sum_{j=1}^{a} \left(q^{j-1}[a+b-1-2(j-1)]\right) + q^b(1+q+\cdots+q^{a-1}) \\
&= \sum_{j=1}^{a} \left(q^{j-1}[a+b-1-2(j-1)] + q^b q^{a-1-(j-1)}\right) \\
&= \sum_{j=1}^{a} \left(q^{j-1}[a+b-2(j-1)]\right)
\end{aligned}
$$

□

3.2 Matching Tail End Coefficients

The manual method of finding decompositions is based on finding the eigenvalues of $\wedge^d(V_n)$ given that the eigenvalues of V_n are $\{n, n-2, \cdots, -n\}$. By convention,

$$H(v_1 \wedge ... \wedge v_d) = \sum_{i=1}^{d} v_1 \wedge ...H(v_i)... \wedge v_d.$$

This leads to the following formula.

Theorem 3.5. *If the n distinct eigenvalues of V are $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ for an operator H, then the eigenvalues of $\wedge^d V$ are in the multiset $\{\alpha_{a_1} + \alpha_{a_2} + \cdots + \alpha_{a_d}\}$ where $1 \leq a_1 < a_2 < \cdots < a_r \leq n$.*

Proof. Let U_{α_i} be the set of eigenvectors v where $H(v) = \alpha_i v$. For $v_i \in U_{\alpha_i}$ 0, the set $\{v_1, v_2, \cdots, v_n\}$ is linearly independent and form a basis of V.

Now for $v_{a_1} \wedge v_{a_2} \wedge \cdots \wedge v_{a_d}$ to be non-zero, $a_i \neq a_j$ if $i \neq j$. Due to $v_{a_2} \wedge v_{a_1} = -v_{a_1} \wedge v_{a_2}$, without loss of generality, let $a_1 < a_2 < \cdots < a_d$. Then

$$
\begin{aligned}
H(v_1 \wedge ... \wedge v_d) &= \sum_{i=1}^{d} v_{a_1} \wedge ...H(v_{a_i})... \wedge v_{a_d} \\
&= \sum_{i=1}^{d} \alpha_{a_i}(v_{a_1} \wedge ...v_{a_i}... \wedge v_{a_d}) \\
&= (\alpha_{a_1} + \cdots + \alpha_{a_d})(v_{a_1} \wedge ...v_{a_i}... \wedge v_{a_d}),
\end{aligned}
$$

thus confirming that $\alpha_{a_1} + \cdots + \alpha_{a_d}$ is an eigenvalue.

For completeness, the number of elements in the multiset is $\binom{n}{r}$ via the number of ways to choose a set of r distinct positive whole numbers from 1 to n, and the number of ways to pick r distinct eigenvectors from $\{v_1, \cdots, v_n\}$ corresponding to the eigenvalues.

□

Manually considering the eigenvalues for $\wedge^d(V_n)$ allows for the use of two functions used as an aid to finding the decomposition.

- Let $S(d, n, a)$ be the number of ways to choose d different elements from $\{-n, -n+2, \cdots, n-2, n\}$ to sum to the number a.
- Let $C(d, n, a) = S(d, n, a) - S(d, n, a+2)$.

For each subset in $\{\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_r}\}$ that sums to a, multiplying each eigenvalue by -1 results in a set of eigenvalues that sum to $-a$ because of the symmetry in $\{-n, -n+2, \cdots, n-2, n\}$. Because of the bijection, $S(d, n, a) = S(d, n, -a)$. Based on the visualization for eigenvalues of the direct sum $\oplus V_\alpha$, for $a \geq 0$, the value of $C(d, n, a)$ is the number of appearances (multiplicity) of V_a in the expansion of $\wedge^d(V_n)$. [4, p. 151].

Using the $S(d, n, a)$ and $C(d, n, a)$ allows for a recursive relation that can be used to compute terms of the expansion of $\wedge^d(V_n)$. The following goes into some of the base cases that can be directly computed.

Theorem 3.6 (Base Case Theorem 1). *For some starting base cases,*

- $\wedge^n V_{n-1} = V_0$
- $\wedge^n V_n = V_n$
- $\wedge^n V_{n+k} = \wedge^{k+1} V_{n+k}$

Proof. For $\wedge^n V_{n-1}$, the eigenvalues of V_{n-1} are the numbers of the same parity from $n-1$ to $-n+1$ inclusive, which are n numbers. All of them have to be selected and they all sum to 0.

For $\wedge^n V_n$, the eigenvalues of V_n are the numbers of the same parity from n to $-n$ inclusive, and all the $n+1$ numbers sum to 0 by symmetry. So summing n different numbers would mean leaving out each element from n to $-n$, and subtracting each sum would lead to the eigenvalues of $\wedge^V_n$ being the numbers of the same parity from n to $-n$.

For $\wedge^n V_{n+k}$, the eigenvalues of V_{n+k} are the numbers of the same parity from $n+k$ to $-n-k$ inclusive, and there are $n+k+1$ such numbers. By symmetry, all of the $n+k+1$ numbers sum to 0, so for each set of n numbers that sum to a, the remaining $k+1$ numbers sum to $-a$. But if we flip the sign of those remaining $k+1$ numbers, then those numbers sum to a. This means that there is a bijection between choosing n numbers that sum to a and $k+1$ numbers that sum to a. □

Theorem 3.7 (Base Case Theorem 2). $C(d, n, a) = 0$ *if* $a > dn - d(d-1)$.

Proof. The largest d elements of the set $\{n, n-2, \cdots, -n\}$ is $n, n-2, \cdots, n-2(d-1)$. These elements have a sum of $\frac{1}{2} \cdot (2n - 2d + 2) \cdot d = dn - d(d-1)$. Therefore, if $a > dn - d(d-1)$, then it is impossible to select d elements of the set $\{n, n-2, \cdots, -n\}$ such that the sum is a. As a result, $S(d, n, a) = S(d, n, a+2) = 0$, so $C(d, n, a) = 0$. □

Theorem 3.8 (Base Case Theorem 3). $C(d, n, -a) = -C(d, n, a-2)$.

Proof. First, $C(d, n, a)$ is defined as equalling $S(d, n, a) - S(d, n, a+2)$. Additionally, by symmetry, $S(d, n, a) = S(d, n, -a)$. Therefore,

$$\begin{aligned} C(d, n, -a) &= S(d, n, -a) - S(d, n, -a+2) \\ &= S(d, n, a) - S(d, n, a-2) \\ &= -(S(d, n, a-2) - S(d, n, a)) \\ &= -C(d, n, a-2). \end{aligned}$$

□

In addition to the base cases, I will be working with the following recurrence relation.

Theorem 3.9 (C(d,n,a) Theorem 1).

$$C(d, n+1, a+d) = C(d, n, a) + C(d-1, n, a+n+2)$$

if $d \geq 2$.

Proof. The value of $S(d, n+1, a+d)$ is the number of ways to choose d elements from the set $\{-n-1, -n+1, \cdots, n-1, n+1\}$. We could either include $-n-1$ or not.

- If we do not include $-n-1$, then we are choosing d elements from the set $\{-n+1, \cdots, n-1, n+1\}$ that add up to $a+d$. There is a bijection with choosing d elements from the set $\{-n, -n+2, \cdots, n-2, n\}$ that sum to a because whatever element we pick, we just add 1 to each of the d elements.
- If we do include $-n-1$, then we are choosing $d-1$ elements from the set $\{-n+1, \cdots, n-1, n+1\}$ that add up to $a+d+n+1$. There is a bijection with choosing $d-1$ elements from the set $\{-n, -n+2, \cdots, n-2, n\}$ that sum to $a+n+2$ because whatever element we pick, we just add 1 to each of the $d-1$ elements.

And the definition $C(n, a, b) = S(n, a, b) - S(n, a, b+2)$ could be used to obtain

$$C(d, n+1, a+d) = C(d, n, a) + C(d-1, n, a+n+2).$$

□

Notably, the property that $C(d,n,a) = 0$ if $a > dn - d(d-1)$ indicates that all V_a with a non-zero coefficient in the expansion of $\wedge^d(V_n)$ must satisfy $a \leq dn - (d-1)d$, allowing for a starting point to be large enough values of a for $C(d,n,a)$. This means that the recurrence relation can be used to identify values of $C(d,n,a)$ where a is close to $dn - d(d-1)$, regardless of the value of d.

Theorem 3.10 (C(d,n,a) Theorem 2)**.** *If $a > (d-1)n-(d-1)(d-2)-n-2$, then $C(d,n+1,a+d) = C(d,n,a)$. This essentially means that*

$$C(d,n+1,d(n+1)-d(d-1)-k) = C(d,n,dn-d(d-1)-k)$$

if $k < -2d+2n+4$.

Proof. By Base Case Theorem 2, if $a > dn-(d-1)d$, then $C(d,n,a) = 0$. Then using the $C(d,n+1,a+d) = C(d,n,a)+C(d-1,n,a+n+2)$ and setting $a > (d-1)n-(d-1)(d-2)-n-2$ would make $C(d-1,n,a+n+2)$ equal to zero. Then substituting $dn - d(d-1) - k$ for a would finish off the proof. $\square$

Notably, when doing $\wedge^d(V_n)$, we want $n \geq d-1$. If n is at most $d-2$, then coefficients come from selecting d elements from the set $\{-d+2, -d+4, \cdots, d-4, d-2\}$, which would only have $d-1$ elements. That would be impossible. So when $n \geq d-1$ in the cases that we care about, we get $2n - 2d + 4 \geq 2 > 0$.

This essentially results in corresponding equivalent coefficients for $\wedge^d(V_n)$ and $\wedge^d(V_{n+1})$ at the very end, as in the a in V_a is large. This property is observable in the LiE code below by comparing some of the tail end coefficients, like 7X[35] and 7X[30] in the top two rows and 5X[24] and 5X[20] in the bottom two rows. This allows for an approach to determining coefficients of $V_{dn-d(d-1)-2k}$ in the expansion of $\wedge^d(V_n)$ for small enough k.

```
> alt_tensor(5,X[15],A1)
      2X[ 1] + 4X[ 3] + 6X[ 5] + 8X[ 7] + 9X[ 9] +11X[11] +11X[13] +
     13X[15] +12X[17] +13X[19] +12X[21] +12X[23] +11X[25] +11X[27] +
      9X[29] + 9X[31] + 7X[33] + 7X[35] + 5X[37] + 5X[39] + 3X[41] +
      3X[43] + 2X[45] + 2X[47] + 1X[49] + 1X[51] + 1X[55]
> alt_tensor(5,X[14],A1)
      4X[ 2] + 3X[ 4] + 7X[ 6] + 6X[ 8] +10X[10] + 8X[12] +11X[14] +
      9X[16] +11X[18] + 9X[20] +10X[22] + 8X[24] + 9X[26] + 6X[28] +
      7X[30] + 5X[32] + 5X[34] + 3X[36] + 3X[38] + 2X[40] + 2X[42] +
      1X[44] + 1X[46] + 1X[50]
> alt_tensor(4,X[14],A1)
     2X[ 0] +4X[ 4] +2X[ 6] +5X[ 8] +3X[10] +6X[12] +4X[14] +6X[16] +
     4X[18] +6X[20] +4X[22] +5X[24] +3X[26] +4X[28] +2X[30] +3X[32] +
     1X[34] +2X[36] +1X[38] +1X[40] +1X[44]
> alt_tensor(4,X[13],A1)
     2X[ 0] +4X[ 4] +1X[ 6] +5X[ 8] +3X[10] +5X[12] +3X[14] +6X[16] +
     3X[18] +5X[20] +3X[22] +4X[24] +2X[26] +3X[28] +1X[30] +2X[32] +
     1X[34] +1X[36] +1X[40]
```

3.3 Connection with Integer Partitions

First I show a bijection to integer partitions with an upper bound for each element, as I can then use already-established notation as defined by MJ Kronenberg. [3]

Theorem 3.11 (Partition Bijection 1)**.** *The value of $S(d,n,dn-d(d-1)-2k)$ is equal to $Q(k+\frac{d(d+1)}{2}, d, n+1)$, which is the number of partitions of $k + \frac{d(d+1)}{2}$ into exactly d parts that are each at most $n+1$.*

Proof. Let A be the set of all sets of d distinct integers from $\{-n, -n+2, \cdots, n-2, n\}$ that add to a, and let B be the set of all sets of d distinct integers from $\{1, 2, \cdots, n+1\}$ that add to $\frac{a+n+2}{2}$. The following shows a bijection between A and B.

- For each set in A, each element can be multiplied by -1, added by $n+2$, and then divided by 2. The sum of the elements after the transformation would be $\frac{2k-dn+d(d-1)+n+2}{2} = k + \frac{d(d+1)}{2}$.
- For each set in B, each element can be multiplied by 2, subtracted by $n+2$, and then multiplied by -1. The sum of the elements after the transformation would be $-(2k+d(d+1)-d(n+2)) = dn-d(d-1)-2k$.

As a result, the number of elements in A equals the number of elements in B, and so by using the definitions,

$$S(d,n,dn-d(d-1)-2k) = Q(k+\frac{d(d+1)}{2}, d, n+1).$$

$\square$

This bijection allows for computation of $C(d,n,a)$ when a is close to $dn-d(d-1)$, allowing for generalizing the "tail end" coefficients.

Theorem 3.12 (C(d,n,dn - d(d-1) - 2k) Theorem 1)**.** *If $k \leq -d+n+1$, then $C(d,n,dn-d(d-1)-2k)$ equals $P^{\#}(k,2,d)$, which is the number of integer partitions of k into elements of size at least 2 and at most d.*

Here's a lemma focusing on integer partitions with notation from MJ Kronenberg.

Lemma 3.13. *Using the terms $P(n,m)$ and $P^{\#}(n,p_{min},p_{max})$ as defined by MJ Kronenberg [3],*

$$Q(n+\frac{d(d+1)}{2}, d) - Q(n-1+\frac{d(d+1)}{2}, d) = P^{\#}(n,2,d).$$

Proof. Because $Q(n+\frac{d(d+1)}{2}, d)$ is the number of partitions of $n+\frac{d(d+1)}{2}$ into exactly d distinct elements, by using the "staircase bijection" (subtract 0 from smallest, subtract 1 from second smallest, etc.), the value of $Q(n+\frac{d(d+1)}{2}, d)$ is equal to the number of partitions of $n+\frac{d(d+1)}{2}-\frac{d(d-1)}{2} = n+d$ into exactly d elements, which is the number of partitions of $n+d$ where the largest element is d. Since each partition must have an element of size d, we then have the number of partitions of n where the elements are at most d, or $P^{\#}(n,1,d)$.

By using similar reasoning, $Q(n-1+\frac{d(d+1)}{2}, d) = P^{\#}(n-1,1,d)$. Now

$$\sum_{j=1}^{n}\left(P^{\#}(j,2,d)\right) = P^{\#}(n,1,d)$$

because for each partition of j that has elements from 2 to d, there is only 1 way to insert some number of ones to get a sum of n. Therefore,

$$\begin{aligned} P^{\#}(n,1,d) - P^{\#}(n-1,1,d) &= \sum_{j=1}^{n}\left(P^{\#}(j,2,d)\right) - \sum_{j=1}^{n-1}\left(P^{\#}(j,2,d)\right) \\ &= P^{\#}(n,2,d). \end{aligned}$$

$\square$

This lemma will be used for the rest of the proof.

Proof. From the Partition Bijection 1,

$$S(d,n,dn-d(d-1)-2k) = Q(k+\frac{d(d+1)}{2}, d, n+1).$$

If the value of $k+\frac{d(d+1)}{2}$ is at most $1+2+\cdots+d-1+n+1 = \frac{d(d-1)}{2}+n+1$, then the size restriction to elements being at most $n+1$ does not matter due to having no possibilities if one element is $n+2$ or larger. This happens if $k \leq -d+n+1$, and in that case, $Q(k+\frac{d(d+1)}{2}, d, n+1) = P(k+\frac{d(d+1)}{2}, d)$, which results in

$$S(d,n,dn-d(d-1)-2k) = Q(k+\frac{d(d+1)}{2}, d)$$

$$S(d,n,dn-d(d-1)-2(k-1)) = Q(k-1+\frac{d(d+1)}{2}, d)$$

and, by taking the difference and using the Lemma,

$$\begin{aligned} C(d,n,dn-d(d-1)-2k) &= Q(k+\frac{d(d+1)}{2}, d) - Q(k-1+\frac{d(d+1)}{2}, d) \\ &= P^{\#}(k,2,d) \end{aligned}$$

if $k \leq -d+n+1$. $\square$

As an example of this theorem in action, for the coefficient of V_{35} in the expansion of $\wedge^5(V_{15})$, since $(15 \cdot 5 - 20) - 35 = 20$ and $10 \leq -5 + 15 + 1$, the coefficient of V_{35} equals the number of ways to split 10 into a partition where elements are at least 2 and at most 5, which would be the coefficient of x^{10} in the generating series $\frac{1}{\prod_{i=2}^{5}(1-x^i)}$, which would be 7.

This direct generalization allows for the use of generating functions to compute coefficients, but if $k > -d+n+1$ for $C(d, n, dn - d(d-1) - 2k)$, then accounting for overcounting needs to be done. The next few theorems do some overcounting corrections that allow for computation of $C(d, n, dn - d(d-1) - 2k)$ for some values of k where $k > -d + n + 1$.

Theorem 3.14 (Partition Bijection 2)**.** *Let X be the set of integer partitions of $k + \frac{d(d+1)}{2}$ with d distinct elements such that if a partition has $p_{max} > n + 2$ as the largest element, then that partition has the element $p_{max} - 1$. Let Y be the set of integer partitions of $k + \frac{d(d+1)}{2}$ with d distinct elements such that $p_{max} = n + 2$. Then the value of*

$$\sum_{j=n+2}^{\infty} Q(k + \frac{d(d+1)}{2} - j, d-1, j-1) - \sum_{j=n+2}^{\infty} Q(k - 1 + \frac{d(d+1)}{2} - j, d-1, j-1)$$

is equal to $|X| + |Y|$.

The following lemma will be used.

Lemma 3.15. *Let A be the set of integer partitions of $k + \frac{d(d+1)}{2}$ with d distinct elements that have at least one element at least $n + 2$, and let B be the set of integer partitions of $k - 1 + \frac{d(d+1)}{2}$ with d distinct elements that have at least one element at least $n + 2$. The set A can be partitioned into three subsets.*

- *$A_1 \subset A$ is the set of all partitions such that if a partition has $p_{max} > n + 2$ as the largest element, then that partition does not have the element $p_{max} - 1$.*
- *$A_2 \subset A$ is the set of all partitions such that if a partition has $p_{max} > n + 2$ as the largest element, then that partition has the element $p_{max} - 1$.*
- *$A_3 \subset A$ is the set of all partitions that have $p_{max} = n + 2$.*

Then there is a bijection between B and A_1.

Proof. For each partition in B, let $p_{\max}$ be the largest element. Since this partition has no duplicate elements, adding one to the largest element would result in a partition in A_1 because the second largest element must be less than $p_{\max} - 1$.

For each partition in A_1, subtract 1 to the largest element. Since $p_{\max} - 1$ is not in that partition, the result would be in B as the partition would still have distinct elements. This completes the bijection. ☐

This lemma would be used later in the rest of the proof.

Proof. By Partition Bijection 1,

$$S(d, n, dn - d(d-1) - 2k) = Q(k + \frac{d(d+1)}{2}, d, n+1) = P(k + d, d, n - d + 2).$$

Due to the size limit, there can not be partitions of $\{1, 2, \cdots, n+1\}$ that have elements of size $n+2$ or higher. As a result, the value of $S(d, n, dn - d(d-1) - 2k)$ is

$$P(k + d, d) - \sum_{j=n+2}^{\infty} Q(k + \frac{d(d+1)}{2} - j, d-1, j-1)$$

where the summation represents all "illegal" partitions.
Now the value of $S(d, n, dn - d(d-1) - 2k)$ is

$$P(k + d, d) - \sum_{j=n+2}^{\infty} Q(k + \frac{d(d+1)}{2} - j, d-1, j-1)$$

and the value of $S(d, n, dn - d(d-1) - 2(k-1))$ is

$$P(k - 1 + d, d) - \sum_{j=n+2}^{\infty} Q(k - 1 + \frac{d(d+1)}{2} - j, d-1, j-1)$$

and so the value of

$$\sum_{j=n+2}^{\infty} Q(k+\frac{d(d+1)}{2}-j,d-1,j-1)-\sum_{j=n+2}^{\infty} Q(k-1+\frac{d(d+1)}{2}-j,d-1,j-1)$$

is equal to $|A_2|+|A_3|=|X|+|Y|$ by the lemma. As a result,

$$\begin{aligned}C(d,n,dn-d(d-1)-2k) &= P(k+d,d)-P(k-1+d,d)-(|X|+|Y|)\\ &= P^{\#}(k,2,d)-(|X|+|Y|)\end{aligned}$$

and so the question of coefficients reduces to the question of finding $|X|+|Y|$. □

Theorem 3.16 (C(d,n,dn - d(d-1) - 2k) Theorem 2)**.** *If $k\le 2(-d+n+2)$, then*

$$C(d,n,dn-d(d-1)-2k)=P^{\#}(k,2,d)-P^{\#}(k+d-(n+2),1,d-1)$$

and the formula can be considered an extension of the previous theorem.

Proof. From Partition Bijection 2, the goal is to count the partitions with d distinct elements of $k+\frac{d(d+1)}{2}$ that either have the largest element equal to $n+2$ or have the largest element greater than $n+2$ and the second largest element differ by 1 from the largest element. There are no cases where the largest term is greater than $n+2$ and the second largest term is one less than the smaller term. The value of $(1+\cdots+d-2)+n+2+n+3$ is $\frac{(d-1)(d-2)}{2}+2n+5$, and all partitions with d elements will have sums at least that. However, the value of $-2d+2n+4+\frac{d(d+1)}{2}$ is $\frac{d(d-3)}{2}+2n+4$, which is smaller.

Therefore, the only cases to count are the ones where the largest term equals $n+2$. All $d-1$ remaining terms must be at most $n+1$ because otherwise, the remaining $d-2$ terms must sum to at most $-2d+\frac{d(d+1)}{2}$, but the sum of the first $d-2$ positive whole numbers is $\frac{(d-1)(d-2)}{2}$.

In other words,

$$\sum_{j=n+2}^{\infty} Q(k+\frac{d(d+1)}{2}-j,d-1,j-1)-\sum_{j=n+2}^{\infty} Q(k-1+\frac{d(d+1)}{2}-j,d-1,j-1)$$

is equal to the number of partitions of $k+\frac{d(d+1)}{2}-(n+2)$ parts into $d-1$ distinct elements, which is the number of ways to partition $k+d-(n+2)$ into at most $d-1$ elements, which is the number of ways to partition $k+d-(n+2)$ into blocks that have a size of at most $d-1$. Then

$$C(d,n,dn-d(d-1)-2k)=P^{\#}(k,2,d)-P^{\#}(k+d-(n+2),1,d-1).$$

□

As a result, for $0\le k\le 2(-d+n+2)$, the coefficient of $V_{dn-d(d-1)-2k}$ is the coefficient of x^k in the expansion of the generating series

$$\frac{1}{\prod_{i=2}^{d}(1-x^i)}-\frac{x^{n+2-d}}{\prod_{i=1}^{d-1}(1-x^i)}.$$

Although this formula only works for some of the values of k for $C(d,n,dn-d(d-1)-2k)$, this is sufficient for all terms in the expansion of $\wedge^3(V_n)$ and all but one term of $\wedge^4(V_n)$ as $k\le\frac{dn-d(d-1)}{2}$ in order for $dn-d(d-1)-2k$ to be greater than zero and thus have $C(d,n,dn-d(d-1)-2k)$ represent the coefficient of $V_{dn-d(d-1)-2k}$ in the expansion of $\wedge^d(V_n)$.

The steps that account for overcounting imply that the complete generalization would involve an alternating sum, which can help identify the generating function discussed in the next section.

3.4 Complete Generalization

I worked with generating series and substitution to get the complete generalization for the value of $C(d,n,dn-d(d-1)-2k)$.

Theorem 3.17 (C(d,n,dn - d(d-1) - 2k) Generalization)**.** *The value of $C(d,n,dn-(d-1)-2k)$ is the coefficient of x^k in the summation*

$$\left(\sum_{j=1}^{d}\frac{(-1)^{j+1}x^{(j-1)(n-d)+0.5j^2+0.5j-1}}{\prod_{i=1}^{(j-1)}(1-x^i)\prod_{i=2}^{d-(j-1)}(1-x^i)}\right)+\frac{(-1)^d x^{d(n-d)+0.5d(d+3)}}{\prod_{i=2}^{d}(1-x^i)}$$

The rest of the section will be based on proving this theorem by defining

$$F_{n,d}(x) = \sum_{k\geq 0} C(d,n,dn-d(d-1)-2k)x^k$$

and use this to create a recurrence relation, demonstrate the base cases, and then finally show that the recurrence relation holds for the giant expression.

Lemma 3.18.

$$F_{n+1,d}(x) = F_{n,d}(x) + x^{n-d+2}F_{n,d-1}(x).$$

Proof. First, from $C(d,n,a)$ Theorem 1, $C(d,n+1,a+d) = C(d,n,a) + C(d-1,n,a+n+2)$. By multiplying both sides by x^k to the identity shown in $C(d,n,a)$ Theorem 1 and considering the generating series,

$$F_{n+1,d}(x) = F_{n,d}(x) + \sum_{k\geq 0}\left(C(d-1,n,(d-1)n-(d-1)(d-2)-2(k+d-n-2))x^k\right).$$

Now consider the summation $\sum_{k\geq 0}(C(d-1,n,(d-1)n-(d-1)(d-2)-2(k+d-n-2))x^k)$. Factoring x^{n-d+2} yields

$$x^{n-d+2}\sum_{k\geq 0}(C(d-1,n,(d-1)n-(d-1)(d-2)-2(k+d-n-2))x^{k+d-n-2}).$$

Then let $j = k+d-n-2$. Then $k = j+n+2-d$ and the summation is

$$x^{n-d+2}\sum_{j\geq d-n-2}(C(d-1,n,(d-1)n-(d-1)(d-2)-2j)x^j).$$

By Base Case Theorem 1, for $C(d-1,n,(d-1)n-(d-1)(d-2)-2j)=0$ to be non-zero, $d-1\geq n-1$, so $n > d-2$. This means $0 > d-n-2$, and from Base Case Theorem 2, the giant summation is equal to

$$\sum_{j\geq 0}(C(d-1,n,(d-1)n-(d-1)(d-2)-2j)x^j).$$

This expression is equivalent to $F_{n,d-1}(k)$ by definition, so

$$F_{n+1,d}(x) = F_{n,d}(x) + x^{n-d+2}F_{n,d-1}(x).$$

□

We are now ready to show the value of $F_{n,d}(k)$ by using induction with $F_{d-1,d}(x)$ as the base case. For ease of notation, define

$$G_{n,d}(x) = \sum_{j=1}^{d}\left(\frac{(-1)^{j+1}x^{(j-1)(n-d)+0.5j^2+0.5j-1}}{\prod_{i=1}^{(j-1)}(1-x^i)\prod_{i=2}^{d-(j-1)}(1-x^i)}\right).$$

$$H_{n,d}(x) = \frac{(-1)^{d+2}x^{d(n-d)+0.5(d+1)^2+0.5(d+1)-1}}{\prod_{i=2}^{d}(1-x^i)}$$

and so the goal is to show that $F_{n,d}(x) = G_{n,d}(x) + H_{n,d}(x)$.

Proof. First we show

$$F_{d-1,d}(k) = G_{d-1,d}(x) + H_{d-1,d}(x) = 1-x$$

to confirm the base case holds.

By using the q-Binomial Theorem,

$$\begin{aligned}\prod_{j=1}^{d}(1-q^j) &= \sum_{j=0}^{d}\begin{bmatrix} d\\ j\end{bmatrix}_q (-1)^j q^{\frac{j(j+1)}{2}}\\ &= \sum_{j=0}^{d}\frac{\prod_{i=d-(j-1)}^{d}(1-x^i)}{\prod_{i=1}^{j}(1-x^i)}(-1)^j q^{\frac{j(j+1)}{2}}.\end{aligned}$$

Splitting the summation and shifting the index on the right hand side yields

$$\begin{aligned}\prod_{j=1}^{d}(1-x^j) &= \sum_{j=0}^{d-1}\left(\frac{\prod_{i=d-(j-1)}^{d}(1-x^i)}{\prod_{i=1}^{j}(1-x^i)}(-1)^j x^{\frac{j(j+1)}{2}}\right)+(-1)^d x^{\frac{d(d+1)}{2}}\\ &= \sum_{j=1}^{d}\left(\frac{\prod_{i=d-(j-2)}^{d}(1-x^i)}{\prod_{i=1}^{j-1}(1-x^i)}(-1)^{j-1} x^{\frac{(j-1)j}{2}}\right)+(-1)^d x^{\frac{d(d+1)}{2}}.\end{aligned}$$

Dividing both sides by $\prod_{j=2}^{d}(1-x^j)$ results in

$$\begin{aligned}1-x &= \sum_{j=1}^{d}\left(\frac{(-1)^{j-1}x^{\frac{(j-1)j}{2}}}{\prod_{i=1}^{j-1}(1-x^i)\prod_{i=2}^{d-1-(j-2)}(1-x^i)}\right)+\frac{(-1)^d x^{\frac{d(d+1)}{2}}}{\prod_{i=2}^{d}(1-x^i)}\\ &= \sum_{j=1}^{d}\left(\frac{(-1)^{j-1}x^{\frac{(j-1)j}{2}}}{\prod_{i=1}^{j-1}(1-x^i)\prod_{i=2}^{d-(j-1)}(1-x^i)}\right)+\frac{(-1)^d x^{\frac{d(d+1)}{2}}}{\prod_{i=2}^{d}(1-x^i)}\\ &= G_{d-1,d}(x)+H_{d-1,d}(x).\end{aligned}$$

Finally, $F_{d-1,d}(x) = C(d,d-1,0)+xC(d,d-1,-2) = 1-x$, and so the base case holds.
As for the inductive step, the goal is to show that

$$\sum_{j=1}^{d}\left(\frac{(-1)^{j+1}x^{(j-1)(n-d)+0.5j^2+0.5j-1}}{\prod_{i=1}^{(j-1)}(1-x^i)\prod_{i=2}^{d-(j-1)}(1-x^i)}\right)+\frac{(-1)^{d+2}x^{d(n-d)+0.5(d+1)^2+0.5(d+1)-1}}{\prod_{i=2}^{d}(1-x^i)}$$

satisfies the recurrence relation $F_{n+1,d}(k) = F_{n,d}(k)+x^{n-d+2}F_{n,d-1}(k)$, as outlined in the lemma.

First, the value of $x^{n-d+2}F_{n,d-1}(k)$ is

$$x^{n-d+2}\left(\sum_{j=1}^{d-1}\left(\frac{(-1)^{j+1}x^{(j-1)(n-(d-1))+0.5j^2+0.5j-1}}{\prod_{i=1}^{(j-1)}(1-x^i)\prod_{i=2}^{(d-1)-(j-1)}(1-x^i)}\right)+\frac{(-1)^{(d-1)+2}x^{(d-1)(n-(d-1))+0.5(d-1+1)^2+0.5(d-1+1)-1}}{\prod_{i=2}^{d-1}(1-x^i)}\right),$$

which simplifies to

$$\sum_{j=1}^{d-1}\left(\frac{(-1)^{j+1}x^{-dj+0.5j^2+jn+1.5j}}{\prod_{i=1}^{(j-1)}(1-x^i)\prod_{i=2}^{(d-1)-(j-1)}(1-x^i)}\right)+\frac{(-1)^{(d-1)+2}x^{-0.5d^2+dn+1.5d}}{\prod_{i=2}^{d-1}(1-x^i)}.$$

Shifting the index leads to

$$\sum_{j=2}^{d}\left(\frac{(-1)^{j}x^{-d(j-1)+0.5(j-1)^2+(j-1)n+1.5(j-1)}}{\prod_{i=1}^{(j-2)}(1-x^i)\prod_{i=2}^{(d-1)-(j-2)}(1-x^i)}\right)+\frac{(-1)^{(d-1)+2}x^{-0.5d^2+dn+1.5d}}{\prod_{i=2}^{d-1}(1-x^i)},$$

which simplifies to

$$\sum_{j=2}^{d}\left(\frac{(-1)^{j}x^{-dj+d+0.5j^2+jn+0.5j-n-1}}{\prod_{i=1}^{(j-2)}(1-x^i)\prod_{i=2}^{(d-1)-(j-2)}(1-x^i)}\right)+\frac{(-1)^{d+1}x^{-0.5d^2+dn+1.5d}}{\prod_{i=2}^{d-1}(1-x^i)}.$$

In the giant summation, multiplying each part by $\frac{1-x^{j-1}}{1-x^{j-1}}$ results in

$$\sum_{j=2}^{d}\left(\frac{(-1)^{j}x^{-dj+d+0.5j^2+jn+0.5j-n-1}(1-x^{j-1})}{\prod_{i=1}^{(j-1)}(1-x^i)\prod_{i=2}^{(d-1)-(j-2)}(1-x^i)}\right)+\frac{(-1)^{d+1}x^{-0.5d^2+dn+1.5d}}{\prod_{i=2}^{d-1}(1-x^i)},$$

which later expands to

$$\sum_{j=2}^{d}\left(\frac{(-1)^{j}x^{-dj+d+0.5j^2+jn+0.5j-n-1}}{\prod_{i=1}^{(j-1)}(1-x^i)\prod_{i=2}^{(d-1)-(j-2)}(1-x^i)}+\frac{(-1)^{j+1}x^{-dj+d+0.5j^2+jn+0.5j-n-1+j-1}}{\prod_{i=1}^{(j-1)}(1-x^i)\prod_{i=2}^{(d-1)-(j-2)}(1-x^i)}\right)+\frac{(-1)^{d+1}x^{-0.5d^2+dn+1.5d}}{\prod_{i=2}^{d-1}(1-x^i)}.$$

By adding this value to $F_{n,d}(k)$, terms cancel and the result is

$$\begin{aligned}&\frac{1}{\prod_{i=2}^{d}(1-x^i)}+\sum_{j=2}^{d}\left(\frac{(-1)^{j+1}x^{-dj+d+0.5j^2+jn+0.5j-n-1+j-1}}{\prod_{i=1}^{(j-1)}(1-x^i)\prod_{i=2}^{(d-1)-(j-2)}(1-x^i)}\right)\\ &+\frac{(-1)^{d+1}x^{-0.5d^2+dn+1.5d}}{\prod_{i=2}^{d-1}(1-x^i)}+\frac{(-1)^{d+2}x^{d(n-d)+0.5(d+1)^2+0.5(d+1)-1}}{\prod_{i=2}^{d}(1-x^i)},\end{aligned}$$

and the summation can be combined to get

$$\sum_{j=1}^{d}\left(\frac{(-1)^{j+1}x^{(j-1)(n+1-d)+0.5j^2+0.5j-1}}{\prod_{i=1}^{(j-1)}(1-x^i)\prod_{i=2}^{(d-1)-(j-2)}(1-x^i)}\right)+\frac{(-1)^{d+1}x^{-0.5d^2+dn+1.5d}}{\prod_{i=2}^{d-1}(1-x^i)}+\frac{(-1)^{d+2}x^{d(n-d)+0.5(d+1)^2+0.5(d+1)-1}}{\prod_{i=2}^{d}(1-x^i)}$$

For the last part, as $\frac{(-1)^{d+1}x^{-0.5d^2+dn+1.5d}}{\prod_{i=2}^{d-1}(1-x^i)}=\frac{(-1)^{d+1}x^{-0.5d^2+dn+1.5d}(1-x^d)}{\prod_{i=2}^{d}(1-x^i)}$ and $\frac{(-1)^{d+2}x^{d(n-d)+0.5(d+1)^2+0.5(d+1)-1}}{\prod_{i=2}^{d}(1-x^i)}=\frac{(-1)^{d+2}x^{1.5d-0.5d^2+dn}}{\prod_{i=2}^{d}(1-x^i)}$, terms cancel and the summation results in

$$\sum_{j=1}^{d}\left(\frac{(-1)^{j+1}x^{(j-1)(n+1-d)+0.5j^2+0.5j-1}}{\prod_{i=1}^{(j-1)}(1-x^i)\prod_{i=2}^{(d-1)-(j-2)}(1-x^i)}\right)+\frac{(-1)^{d+2}x^{-0.5d^2+dn+2.5d}}{\prod_{i=2}^{d}(1-x^i)},$$

which is indeed $F_{n+1,d}(k)$. □

4 Applications

Given enough persistence with generating functions, one can compute the coefficients of $\wedge^d(V_n)$ for any value of d. However, the formula is gigantic, so the goal is to find an easier way to compute $C(d,n,a)$ and identify patterns from that formula, often by finding recurrence relations that can be used to generate explicit formulas.

In the generating series $F_{n,d}(x)$, the coefficient of x^k represents the value of $C(d,n,dn-d(d-1)-2k)$. If $k\le\frac{dn-d(d-1)}{2}$, then because $dn-d(d-1)-2k\ge 0$, the value of $C(d,n,dn-d(d-1)-2k)$ represents the appearances of V_a where $a=dn-d(d-1)-2k$.

Most of the process to finding recurrence relations (and by extension, Gaussian binomial coefficient identities) involving taking finite differences on the value of n. The generating series $F_{n-s,d}(x)$ represents the value of $C(d,n-s,d(n-s)-d(d-1)-2k)$ for x^k. The positive difference of $dn-d(d-1)-2k$ and $d(n-s)-d(d-1)-2k$ is ds, so for a constant d,

$$F_{n,d}(x)-x^{\frac{ds}{2}}F_{n-s,d}(x)$$

is the generating series of $C(d,n,a)-C(d,n-s,a)$ where $a=dn-d(d-1)-2k$. As the denominator of terms in $F_{n,d}(x)$ involve products of $1-x^j$ for some positive integers j, the main strategy to finding recurrence relations (and by extension, Gaussian binomial identities) involves repeatedly taking first differences until the terms are of form either x^j or $\frac{x^j}{1-x}$ where $j\in Z_{\ge 0}$.

Below are some examples of this technique in action, first done on $d=2$ and $d=3$ to show known results and later done on $d=3$ and $d=4$ to show what the generating series can find.

4.1 d = 2

If $d=2$, then $F_{n,2}(x)$ simplifies to

$$\frac{1}{1-x^2}-\frac{x^n}{1-x}+\frac{x^{2n+1}}{1-x^2}.$$

The value of $2n-2-2k$ is non-negative if k is at most $\frac{2n-2}{2}=n-1$. In that case, the coefficient of x^k is 1 if k is even and 0 if k is odd. When setting $a=2n-2-2k$, this means

$$C(2,n,a)=\begin{cases}1, & \text{if } a\ge 0 \text{ and } a\equiv 2n-2 \pmod 4\\ 0, & \text{else}\end{cases},$$

just like what is already known.

4.2 d = 3

If $d=3$, then $F_{n,3}(x)$ simplifies to

$$\frac{1}{(1-x^2)(1-x^3)}-\frac{x^{n-1}}{(1-x)(1-x^2)}+\frac{x^{2n-1}}{(1-x)(1-x^2)}-\frac{x^{3n}}{(1-x^2)(1-x^3)}.$$

The value of $3n-6-2k$ is non-negative if k is at most $\frac{3n-6}{2}$, but $2n-1>\frac{3n-6}{2}$. Therefore, we only need to consider the first two terms in the summation.

4.2.1 $\wedge^3 V_n$ and $\wedge^3 V_{n-2}$

Consider $F_{n,3}(x) - x^3 F_{n-2,3}(x)$, which takes the difference of coefficients from $\wedge^3 V_n$ and $\wedge^3 V_{n-2}$. The generating series $F_{n,3}(x)$ is

$$\frac{1}{(1-x^2)(1-x^3)} - \frac{x^{n-1}}{(1-x)(1-x^2)} + \frac{x^{2n-1}}{(1-x)(1-x^2)} - \frac{x^{3n}}{(1-x^2)(1-x^3)}.$$

and the generating series $x^3 F_{n-2,3}(x)$ is

$$\frac{x^3}{(1-x^2)(1-x^3)} - \frac{x^{n}}{(1-x)(1-x^2)} + \frac{x^{2n-2}}{(1-x)(1-x^2)} - \frac{x^{3n-3}}{(1-x^2)(1-x^3)}.$$

Taking the difference yields

$$\frac{1-x^3}{(1-x^2)(1-x^3)} - \frac{x^{n-1}-x^n}{(1-x)(1-x^2)} + \frac{x^{2n-1}-x^{2n-2}}{(1-x)(1-x^2)} - \frac{x^{3n}-x^{3n-3}}{(1-x^2)(1-x^3)},$$

which simplifies to

$$\frac{1}{1-x^2} - \frac{x^{n-1}}{1-x^2} + \frac{x^{2n-1}-x^{2n-2}}{(1-x)(1-x^2)} - \frac{x^{3n}-x^{3n-3}}{(1-x^2)(1-x^3)}.$$

The $\frac{x^{n-1}}{1-x^2}$ can explain the parity difference seen in the following LiE results, where V_0 seems to alternate between showing up and not showing up for even n while V_3 consistently shows up for odd n.

```
> alt_tensor(3,X[10],A1)
    1X[ 0] +1X[ 4] +1X[ 6] +2X[ 8] +1X[10] +2X[12] +1X[14] +1X[16] +
    1X[18] +1X[20] +1X[24]
> alt_tensor(3,X[12],A1)
    1X[ 2] +2X[ 6] +1X[ 8] +2X[10] +2X[12] +2X[14] +1X[16] +2X[18] +
    1X[20] +1X[22] +1X[24] +1X[26] +1X[30]
> alt_tensor(3,X[14],A1)
    1X[ 0] +1X[ 4] +1X[ 6] +2X[ 8] +1X[10] +3X[12] +2X[14] +2X[16]
    +2X[18] +2X[20] +1X[22] +2X[24] +1X[26] +1X[28] +1X[30] +1X[32] +1X[36]
```

```
> alt_tensor(3,X[11],A1)
    1X[ 3] +1X[ 5] +1X[ 7] +2X[ 9] +2X[11] +1X[13] +2X[15] +1X[17] +
    1X[19] +1X[21] +1X[23] +1X[27]
> alt_tensor(3,X[13],A1)
    1X[ 3] +1X[ 5] +1X[ 7] +2X[ 9] +2X[11] +2X[13] +2X[15] +2X[17] +
    1X[19] +2X[21] +1X[23] +1X[25] +1X[27] +1X[29] +1X[33]
> alt_tensor(3,X[15],A1)
    1X[ 3] +1X[ 5] +1X[ 7] +2X[ 9] +2X[11] +2X[13] +3X[15] +2X[17] +
    2X[19] +2X[21] +2X[23] +1X[25] +2X[27] +1X[29] +1X[31] +1X[33] +1X[35] +1X[39]
```

If n is odd, then $n-1$ and 0 have the same parity, so some terms from the expansion of $\frac{x^{n+1}}{1-x^2}$ cancel with terms from the expansion of $\frac{1}{1-x^2}$. This is not the case if n is even.

4.2.2 $\wedge^3 V_n$ and $\wedge^3 V_{n-4}$

Consider $F_{n,3}(x) - x^6 F_{n-4,3}(x)$, which takes the difference of coefficients from $\wedge^3 V_n$ and $\wedge^3 V_{n+4}$. The generating series $F_{n,3}(x)$ is

$$\frac{1}{(1-x^2)(1-x^3)} - \frac{x^{n-1}}{(1-x)(1-x^2)} + \frac{x^{2n-1}}{(1-x)(1-x^2)} - \frac{x^{3n}}{(1-x^2)(1-x^3)}.$$

and the generating series $x^6 F_{n-4,3}(x)$ is

$$\frac{x^6}{(1-x^2)(1-x^3)} - \frac{x^{n+1}}{(1-x)(1-x^2)} + \frac{x^{2n-3}}{(1-x)(1-x^2)} - \frac{x^{3n-6}}{(1-x^2)(1-x^3)}.$$

Taking the difference yields $C(3, n, 3n-6-2k) - C(3, n-4, 3n-6-2k)$, which is

$$\frac{1-x^6}{(1-x^2)(1-x^3)} - \frac{x^{n-1}-x^{n+1}}{(1-x)(1-x^2)} + \frac{x^{2n-1}-x^{2n-3}}{(1-x)(1-x^2)} - \frac{x^{3n+12}-x^{3n-6}}{(1-x^2)(1-x^3)},$$

which simplifies to

$$\frac{1+x^3}{1-x^2} - \frac{x^{n-1}}{1-x} + \frac{x^{2n-1}-x^{2n-3}}{(1-x)(1-x^2)} - \frac{x^{3n}-x^{3n-6}}{(1-x^2)(1-x^3)}.$$

Now $\frac{1+x^3}{1-x^2}$ is equivalent to the expansion of $1+x^2+\frac{x^3}{1-x}$. So regardless of n being even or odd, the difference observations are consistent. The resulting generating function simplifies to

$$1+\frac{x^2-x^{n-1}}{1-x} + \frac{x^{2n-1}-x^{2n-3}}{(1-x)(1-x^2)} - \frac{x^{3n}-x^{3n-6}}{(1-x^2)(1-x^3)}.$$

When converting to the corresponding q-number identities, the constant term of the generating function corresponds to the coefficient of V_{3n-6}, which corresponds to $[3n-5]$. As $3n-6-2(n-2)=n-2$, the coefficient of x^{n-2} corresponds to the coefficient of V_{n-2}, which corresponds to $[n-1]$. The identity

$$q^3[n+1,3] - q^9[n-3,3] = q^3[3n-5] + q^5[2n-5][n-3]$$

can then be attained.

4.2.3 General Formula

The function $F_{n,3}(x) - x^6F_{n-4,3}(x)$ is the generating series of $C(3,n,a) - C(3,n-4,a)$ for $n \geq 4$. So if $a = 3n-6-2k$ and k is a non-negative integer, for $n \geq 4$,

$$C(3,n,3n-6-2k) - C(3,n-4,3n-6-2k) = \begin{cases} 1, & \text{if } k=0 \\ 0, & \text{if } k=1 \\ 1, & \text{if } 2 \leq k \leq n-2 \\ 0, & \text{if } k > n-2 \end{cases}$$

By substituting $k = \frac{3n-6-a}{2}$, for $n \geq 4$

$$C(3,n,a) - C(3,n-4,a) = \begin{cases} 1, & \text{if } a = 3n-6, \text{ or } n = \frac{a+6}{3} \\ 0, & \text{if } a = 3n-8, \text{ or } n = \frac{a+8}{3} \\ 1, & \text{if } n-2 \leq a \leq 3n-10, \text{ or } \frac{a+10}{3} \leq n \leq a+2 \\ 0, & \text{if } a < n-2, \text{ or } n > a+2 \\ 0, & \text{else} \end{cases}$$

Since the generating series required $n \geq 4$, this means that in addition to $C(3,n,a)$ where $a > 3n-6$, we also need the base cases $C(3,3,a)$ and $C(3,2,a)$. We have

$$C(3,n,a) = \begin{cases} 1, & \text{if } n=2, a=0 \\ 1, & \text{if } n=3, a=3 \\ 0, & \text{else } (2 \leq n \leq 3) \end{cases}$$

Now when determining $C(3,n,a)$, we first consider the smallest possible nonnegative value of j such that $n-4j \leq a+2$ and the largest possible value of j such that $n-4j \geq \frac{a+6}{3}$ for the bounds. This occurs when $\lceil \frac{3n-6-3a}{12} \rceil \leq j \leq \lfloor \frac{3n-6-a}{12} \rfloor$. By using the recursive formula and accounting for $n \equiv \frac{a+8}{3} \pmod 4$, if $a \leq 3n-6$,

$$C(3,n,a) = \begin{cases} \lfloor \frac{3n-6-a}{12} \rfloor - \max(0, \lceil \frac{3n-6-3a}{12} \rceil) + 1, & \text{if } n \not\equiv \frac{a+8}{3} \pmod 4 \\ \lfloor \frac{3n-6-a}{12} \rfloor - \max(0, \lceil \frac{3n-6-3a}{12} \rceil), & \text{if } n \equiv \frac{a+8}{3} \pmod 4 \end{cases}$$

Asymptotic behavior is easier to analyze. Since the generating formula for $C(3,n,3n-6-2k) - C(3,n-4,3n-6-2k)$ has coefficients of x^k equal to zero for large values of k, the value of $C(3,n,a)$ has an upper bound for a set value of a. More specifically, if $n = a+2$, then $C(3,a+2+4j,a) = C(3,a+2,a)$ for nonnegative values of j from the recursive formula. Then by checking when $\frac{a+8}{3}$ is an integer and $a+2 \equiv \frac{a+8}{3} \pmod 4$, we get

$$\max(\{C(3,n,a), n \geq 0\}) = \begin{cases} \lfloor \frac{a}{6} \rfloor + 1, & \text{if } a \not\equiv 1 \pmod 6 \\ \lfloor \frac{a}{6} \rfloor, & \text{if } a \equiv 1 \pmod 6 \end{cases}$$

4.3 d = 4

If $d = 4$, then $F_{n,4}(x)$ simplifies to

$$\frac{1}{\prod_{i=2}^{4}(1-x^i)} - \frac{x^{n-2}}{\prod_{i=1}^{3}(1-x^i)} + \frac{x^{2n-3}}{(1-x)(1-x^2)(1-x^2)} - \frac{x^{3n-3}}{\prod_{i=1}^{3}(1-x^i)} + \frac{x^{4n-2}}{\prod_{i=2}^{4}(1-x^i)}.$$

The value of $4n-12-2k$ is non-negative if k is at most $2n-6$, and $2n-6 < 2n-3$. Therefore, we only need to consider the first two terms in the summation. There are two Q-Binomial Identities that can be obtained from this expression.

4.3.1 First Q-Binomial Identity

The value of $x^4F_{n-2,4}(x)$ simplifies to

$$\frac{x^4}{\prod_{i=2}^{4}(1-x^i)} - \frac{x^{n}}{\prod_{i=1}^{3}(1-x^i)} + \frac{x^{2n-3}}{(1-x)(1-x^2)(1-x^2)} - \frac{x^{3n-5}}{\prod_{i=1}^{3}(1-x^i)} + \frac{x^{4n-6}}{\prod_{i=2}^{4}(1-x^i)}.$$

The value of $x^6F_{n-3,4}(x)$ simplifies to

$$\frac{x^6}{\prod_{i=2}^{4}(1-x^i)} - \frac{x^{n+1}}{\prod_{i=1}^{3}(1-x^i)} + \frac{x^{2n-3}}{(1-x)(1-x^2)(1-x^2)} - \frac{x^{3n-6}}{\prod_{i=1}^{3}(1-x^i)} + \frac{x^{4n-8}}{\prod_{i=2}^{4}(1-x^i)}.$$

The value of $x^{10}F_{n-5,4}(x)$ simplifies to

$$\frac{x^{10}}{\prod_{i=2}^{4}(1-x^i)} - \frac{x^{n+3}}{\prod_{i=1}^{3}(1-x^i)} + \frac{x^{2n-3}}{(1-x)(1-x^2)(1-x^2)} - \frac{x^{3n-8}}{\prod_{i=1}^{3}(1-x^i)} + \frac{x^{4n-12}}{\prod_{i=2}^{4}(1-x^i)}.$$

This means that the value of $(F_{n,4}(x) - x^4F_{n-2,4}(x))$ is

$$\frac{1}{(1-x^2)(1-x^3)} - \frac{x^{n-2}}{(1-x)(1-x^3)} + \cdots$$

and the value of $(x^6F_{n-3,4}(x) - x^{10}F_{n-5,4}(x))$ is

$$\frac{x^6}{(1-x^2)(1-x^3)} - \frac{x^{n+1}}{(1-x)(1-x^3)} + \cdots$$

and so the value of $(F_{n,4}(x) - x^4F_{n-2,4}(x)) - (x^6F_{n-3,4}(x) - x^{10}F_{n-5,4}(x))$ is

$$\frac{1+x^3}{1-x^2} - \frac{x^{n-2}}{1-x} + \cdots = 1 + x^2 + \frac{x^3}{1-x} - \frac{x^{n-2}}{1-x} + \cdots$$

The constant term corresponds to the coefficient of V_{4n-12}, which corresponds to $[4n-11]$. The coefficient of x^{n-3} corresponds to V_{2n-6}, which corresponds to $[2n-5]$. The following identity can then be obtained.

$$\begin{aligned}(q^6 \cdot \begin{bmatrix} n+1 \\ 4 \end{bmatrix} - q^{10} \cdot \begin{bmatrix} n-1 \\ 4 \end{bmatrix}) - (q^{12}\begin{bmatrix} n-2 \\ 4 \end{bmatrix} - q^{16}\begin{bmatrix} n-4 \\ 4 \end{bmatrix}) &= q^6[4n-11] + (q^8[4n-15] + \cdots + q^{n+3}[2n-5]) \\ &= q^6[4n-11] + q^8[3n-10][n-4]\end{aligned}$$

To show the identity does hold, the value of $q^6 \cdot \begin{bmatrix} n+1 \\ 4 \end{bmatrix} - q^{10} \cdot \begin{bmatrix} n-1 \\ 4 \end{bmatrix}$ part is

$$\frac{q^6(1-q^{2n-3})(1-q^{n-1})(1-q^{n-2})}{(1-q^3)(1-q^2)(1-q)}$$

and the value of $q^{12}\begin{bmatrix} n-2 \\ 4 \end{bmatrix} - q^{16}\begin{bmatrix} n-4 \\ 4 \end{bmatrix}$ is

$$\frac{q^{12}(1-q^{2n-9})(1-q^{n-4})(1-q^{n-5})}{(1-q^3)(1-q^2)(1-q)}.$$

Taking the difference and simplifying yields

$$q^6\left(\frac{1-q+q^2+q^{4n-12}-q^{n-2}-q^{3n-8}-q^{4n-11}+q^{4n-10}}{(-1+q)^2}\right)$$

Then then the terms can be grouped as

$$q^6\left(\frac{(1-q)(1-q^{4n-11})}{(1-q)^2}+q^2\frac{(1-q^{n-4})(1-q^{3n-10})}{(1-q)^2}\right).$$

Then as $[a][b]=\frac{(1-q^a)(1-q^b)}{(1-q)^2}$, we get $q^6[4n-11]+q^8[n-4][3n-10]$.

4.3.2 Second Q-Binomial Identity

Since only the first two terms of $F_{n,4}(x)$ need to be considered, the rest of the terms would be hidden for organizational purposes.

The value of $x^{12}F_{n-6,4}(x)$ simplifies to $\frac{x^{12}}{\prod_{i=2}^4(1-x^i)}-\frac{x^{n+4}}{\prod_{i=1}^3(1-x^i)}+\cdots$, the value of $x^2F_{n-1,4}(x)$ simplifies to $\frac{x^2}{\prod_{i=2}^4(1-x^i)}-\frac{x^{n-1}}{\prod_{i=1}^3(1-x^i)}+\cdots$, and the value of $x^{14}F_{n-7,4}(x)$ simplifies to $\frac{x^{14}}{\prod_{i=2}^4(1-x^i)}-\frac{x^{n+5}}{\prod_{i=1}^3(1-x^i)}+\cdots$. This means that the value of $(F_{n,4}(x)-x^{12}F_{n-6,4}(x))$ is

$$\frac{1-x^{12}}{(1-x^2)(1-x^3)(1-x^4)}-\frac{x^{n-2}(1-x^6)}{(1-x)(1-x^2)(1-x^3)}+\cdots$$

and the value of $(x^2F_{n-1,4}(x)-x^{14}F_{n-7,4}(x))$ is

$$\frac{x^2(1-x^{12})}{(1-x^2)(1-x^3)(1-x^4)}-\frac{x^{n-1}(1-x^6)}{(1-x)(1-x^2)(1-x^3)}+\cdots$$

and so the value of $(F_{n,4}(x)-x^{12}F_{n-6,4}(x))-(x^2F_{n-1,4}(x)-x^{14}F_{n-7,4}(x))$ is

$$\frac{(1-x^2)(1-x^{12})}{(1-x^2)(1-x^3)(1-x^4)}-\frac{x^{n-2}(1-x)(1-x^6)}{(1-x)(1-x^2)(1-x^3)}+\cdots.$$

Doing some factorization results in the expression equalling

$$1+x^3+x^4+\frac{x^6-x^{n-2}}{1-x}+x^{n-1}$$

. Converting results in the identity that $q^6\left(\cdot\begin{bmatrix}n+1\\4\end{bmatrix}_q-q^{12}\cdot\begin{bmatrix}n-5\\4\end{bmatrix}_q)-(q^2\begin{bmatrix}n\\4\end{bmatrix}_q-q^{14}\begin{bmatrix}n-6\\4\end{bmatrix}_q)\right)$ equals

$$q^6[4n-11]+q^9[4n-17]+q^{10}[4n-19]+(q^{11}[4n-21]+\cdots+q^{n+2}[2n-5])+q^{n+4}[2n-9],$$

which simplifies to

$$q^6[4n-11]+q^9[4n-17]+q^{10}[4n-19]+q^{11}[3n-13][n-7]+q^{n+4}[2n-7].$$

Now to show that this identity does hold, let $M_1=q^6\left(\cdot\begin{bmatrix}n+1\\4\end{bmatrix}_q-q^{12}\cdot\begin{bmatrix}n-5\\4\end{bmatrix}_q)-(q^2\begin{bmatrix}n\\4\end{bmatrix}_q-q^{14}\begin{bmatrix}n-6\\4\end{bmatrix}_q)\right)$ and $M_2=q^6[4n-11]+q^9[4n-17]+q^{10}[4n-19]+(q^{11}[4n-21]+\cdots+q^{n+2}[2n-5])+q^{n+4}[2n-9]$. Now M_2 can be expanded to

$$-\frac{q^{n+4}}{(1-q)^2}+\frac{q^{n+5}}{(1-q)^2}-\frac{q^{n+6}}{(1-q)^2}-\frac{q^{3n-4}}{(1-q)^2}+\frac{q^{3n-3}}{(1-q)^2}-\frac{q^{3n-2}}{(1-q)^2}+\frac{q^{4n-10}}{(1-q)^2}-\frac{q^{4n-9}}{(1-q)^2}+\frac{q^{4n-7}}{(1-q)^2}-\frac{q^{4n-5}}{(1-q)^2}$$
$$+\frac{q^{4n-4}}{(1-q)^2}+\frac{q^{12}}{(1-q)^2}-\frac{q^{11}}{(1-q)^2}+\frac{q^9}{(1-q)^2}-\frac{q^7}{(1-q)^2}+\frac{q^6}{(1-q)^2},$$

so the goal is to check that M_1 equals that. The value of $\frac{(1-q^4)(1-q^3)(1-q^2)(1-q)}{q^6}M_1$ equals

$$\frac{1}{q^{16}}((-1+q)^2(1+q)^2(1+q^2)(1-q+q^2)(1+q+q^2)(q^{16}-q^{18}+q^{20}+q^{4n}-q^{14+n}-q^{6+3n}-q^{2+4n}+q^{4+4n}))$$

and by clearing common factors, M is equal to

$$\frac{1}{q^{10}} \cdot \frac{1+q^3}{(1-q^2)(1-q)} \cdot (q^{16} - q^{18} + q^{20} + q^{4n} - q^{14+n} - q^{6+3n} - q^{2+4n} + q^{4+4n})),$$

which simplifies to

$$\frac{1-q+q^2}{(1-q)^2} \cdot (q^6 - q^8 + q^{10} + q^{4n-10} - q^{4+n} - q^{3n-4} - q^{4n-8} + q^{4n-6})).$$

As that value is equal to

$$\frac{-q^{n+4}}{(1-q)^2} + \frac{q^{n+5}}{(1-q)^2} - \frac{q^{n+6}}{(1-q)^2} - \frac{q^{3n-4}}{(1-q)^2} + \frac{q^{3n-3}}{(1-q)^2} - \frac{q^{3n-2}}{(1-q)^2} + \frac{q^{4n-10}}{(1-q)^2} - \frac{q^{4n-9}}{(1-q)^2} + \frac{q^{4n-7}}{(1-q)^2} - \frac{q^{4n-5}}{(1-q)^2}$$
$$+ \frac{q^{4n-4}}{(1-q)^2} + \frac{q^{12}}{(1-q)^2} - \frac{q^{11}}{(1-q)^2} + \frac{q^9}{(1-q)^2} - \frac{q^7}{(1-q)^2} + \frac{q^6}{(1-q)^2},$$

we confirm that $M_1 = M_2$ and the identity does hold.

4.3.3 Remarks

Both of the second differences can imply that the second differences asymptotically become zero, resulting in the relation between n and $C(d, n, a)$ asymptotically becoming linear.

5 Conclusion

In summary, the coefficients of V_a in the expansion of $\wedge^d(V_n)$ can be found with a generating series in which the coefficient of x^k is the multiplicity of $V_{dn-d(d-1)-2k}$ for $k \leq \frac{dn-d(d-1)}{2}$. Taking finite differences with this generating series can then be done to identify recurrence relationships with finite differences for the multiplicity of V_a given a fixed d as well as their asymptotic behavior.

The structure of the generating series indicates that repeatedly taking finite differences is a viable strategy for identifying recurrence relationships. However, within the generating series, much of the approach relies on identifying constant terms and terms where the denominator is $1 - x^k$. Therefore, a further area that can be explored include identifying terms where the denominator is the product of two or more values of such $1 - x^{k_i}$.

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