

Math 281A Qualifying Exam – Fall 2026

Name: _____

1. Let $X_1, \dots, X_n$ be i.i.d. $\text{Uniform}(\theta, \theta + 1)$ random variables, where $\theta \in \mathbb{R}$.
 - (a) [8 pts] Find a minimal sufficient statistic for θ .

- (b) [12 pts] Derive the MLE(s) $\hat{\theta}_{\text{MLE}}$ for θ and show they are uniformly asymptotically consistent, in the sense that $|\hat{\theta}_{\text{MLE}} - \theta| \xrightarrow{\mathbb{P}} 0$ as $n \rightarrow \infty$, for all valid choice of MLE.

2. Let $X_1, \dots, X_n$ be i.i.d. random variables with probability density function

$$f(x|\theta, \sigma) = \frac{1}{\sigma} \exp \left[-\frac{(x - \theta)}{\sigma} \right] \mathbb{1}\{x \geq \theta\}, \quad \theta \in \mathbb{R}, \quad \sigma > 0.$$

(a) [10 pts] Suppose σ is known. Find a sufficient and complete statistic for θ .

- (b) [10 pts] Suppose both θ and σ are unknown. Find the MLEs $\hat{\theta}_{\text{MLE}}$ and $\hat{\sigma}_{\text{MLE}}$ for θ and σ , respectively. Prove that $\hat{\theta}_{\text{MLE}}$ and $\hat{\sigma}_{\text{MLE}}$ are independent.

- (c) [10 pts] Derive the asymptotic distribution of $(\hat{\theta}_{\text{MLE}} - \theta)/\hat{\sigma}_{\text{MLE}}$ as $n \rightarrow \infty$, properly scaling the quantity if needed.

Math 281B Qualifying Exam – Fall 2026

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Total points: 40.

1. Let $X_1 \sim \text{Binomial}(n_1, p_1)$ and $X_2 \sim \text{Binomial}(n_2, p_2)$, and assume they are independent. The values of n_1 and n_2 are known, while p_1 and p_2 are unknown parameters.

(i) [8 points] Let $U = X_2$ and $T = X_1 + X_2$. Derive the conditional distribution of $U|T$.

(ii) [8 points] Derive a uniformly most powerful (UMP) test of size α for testing $H_0 : p_1 \geq p_2$ versus $H_1 : p_1 < p_2$.

2. Let $X_1, \dots, X_n$ be i.i.d. from $N(\mu, \theta\mu^2)$, where $\mu > 0$ and $\theta > 0$ are two unknown parameters. Consider a test of size α for testing $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$.

(i) [12 points] Derive the critical region for the likelihood ratio test (LRT). If an exact critical region cannot be determined, use the asymptotic theory of the LRT.

(ii) [12 points] Derive the critical region for Rao's score test.

MATH 281C Qualifying Exam – Fall 2026

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Total points: 30.

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be i.i.d. observations from the regression model

$$Y_i = m(X_i) + \varepsilon_i, \quad i = 1, \dots, n,$$

where X_i has density p on $\mathbb{R}$. Assume that there exists a constant $C_m < \infty$ such that

$$\sup_{x \in \mathbb{R}} |m(x)| \leq C_m.$$

In addition, assume that $\{\varepsilon_i\}_{i=1}^n$ are i.i.d. errors, independent of $\{X_i\}_{i=1}^n$, such that

$$\mathbb{E}(\varepsilon_i) = 0, \quad \mathbb{E}(\varepsilon_i^2) < \infty.$$

Let $K : \mathbb{R} \rightarrow \mathbb{R}$ be such that

$$\int_{\mathbb{R}} K(u) du = 1.$$

In addition, there exists a constant $M_K < \infty$ such that

$$\sup_{u \in \mathbb{R}} |K(u)| \leq M_K,$$

and there exists a constant $L < \infty$ such that

$$|K(u) - K(v)| \leq L|u - v|, \quad u, v \in \mathbb{R}.$$

Fix a bandwidth $h > 0$, not depending on n , and define

$$K_h(t) = \frac{1}{h} K\left(\frac{t}{h}\right).$$

For $x \in [0, 1]$, define

$$\begin{aligned} p_h(x) &= \mathbb{E}\{K_h(x - X_i)\}, \\ r_h(x) &= \mathbb{E}\{K_h(x - X_i)Y_i\}. \end{aligned}$$

Assume that, for some $c_0 > 0$,

$$\inf_{x \in [0, 1]} p_h(x) \geq c_0.$$

Define

$$m_h(x) = \frac{r_h(x)}{p_h(x)}.$$

Define

$$Z_n(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{K_h(x - X_i) \{Y_i - m_h(x)\}}{p_h(x)}, \quad x \in [0, 1].$$

a. [8 pts] Fix $x_1, \dots, x_k \in [0, 1]$. Show that

$$(Z_n(x_1), \dots, Z_n(x_k))$$

converges in distribution to a multivariate normal distribution. Find the covariance matrix of the limiting distribution.

b. [8 pts] Show that there exists a constant $C < \infty$ such that

$$\mathbb{E}\{|Z_n(x) - Z_n(y)|^2\} \leq C|x - y|^2$$

for all $x, y \in [0, 1]$ and all n .

c. [2 pts] Conclude that, as $n \rightarrow \infty$,

$$Z_n \Rightarrow Z_h \quad \text{weakly in } C([0, 1]),$$

where $C([0, 1])$ is equipped with the supremum norm. Find the mean and the covariance function of the Gaussian process Z_h .

d. [12 pts] Define

$$\hat{p}_h(x) = \frac{1}{n} \sum_{i=1}^n K_h(x - X_i),$$

and assume that

$$\sup_{x \in [0, 1]} |\hat{p}_h(x) - p_h(x)| \xrightarrow{\mathbb{P}} 0.$$

Assume that

$$\hat{p}_h(x) > 0 \quad \text{for all } x \in [0, 1].$$

Define

$$\hat{m}_h(x) = \frac{\frac{1}{n} \sum_{i=1}^n K_h(x - X_i) Y_i}{\hat{p}_h(x)}.$$

Show that

$$\sqrt{n}\{\hat{m}_h - m_h\} \Rightarrow Z_h \quad \text{weakly in } C([0, 1]).$$