Name: ____________________

PID: ____________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

1. Write your name on the front page of your exam.
2. Read each question carefully, and answer each question completely.
3. Write your solutions clearly in the exam sheet.
4. Show all of your work; no credit will be given for unsupported answers.
5. You may use the result of one part of the problem in the proof of a later part, even you do not complete the earlier part.

1. (10 points) Suppose $p < q$ are odd primes and G is a group of order $2pq$. Prove that G has two normal subgroups $N_1 \subseteq N_2$ such that $|N_2| = pq$ and $|N_1| = q$.

2. Suppose G is a non-trivial finite solvable group, $|G| = mn$, and $\gcd(m, n) = 1$.

   (a) (5 points) Suppose Q is a minimal normal subgroup of G; that means Q a normal subgroup, $Q \neq \{1\}$ and no proper non-trivial subgroup of Q is normal in G. Prove that Q is an abelian p-group for some prime p.

   (b) (5 points) Prove that G has a subgroup of order m.
   (**Hint**. Use induction and make use of G/Q.)

3. Suppose p is a prime and n is a positive integer. An element $x \in \mathrm{GL}_n(\mathbb{F}_p)$ is called a *p-element* if its order is a power of p.

   (a) (3 points) Prove that x is a p-element if and only if $x - 1$ is nilpotent.

   (b) (7 points) Prove that the number of conjugacy classes of $\mathrm{GL}_n(\mathbb{F}_p)$ that consists of p-elements is the same as the number of conjugacy classes of the symmetric group S_n.

4. Suppose M is a flat A-module.

   (a) (3 points) Prove that, for every ideal I of A, $I \otimes_A M \simeq IM$ with an isomorphism which sends $a \otimes x$ to ax for every $a \in I$ and $x \in M$.

(b) (7 points) For $x_1, \dots, x_n \in M$, suppose

$$a_1 x_1 + \cdots + a_n x_n = 0.$$

Let $I := \langle a_1, \dots, a_n \rangle$ be the ideal generated by a_i's. Let

$$f : A^n \to I, \quad f(b_1, \dots, b_n) := a_1 b_1 + \cdots + a_n b_n,$$

$K := \ker f$, and so

$$0 \to K \to A^n \to I \to 0 \tag{1}$$

is a short exact sequence of A-modules. Prove that there exist $\mathbf{k}_j := (b_{1j}, \dots, b_{nj}) \in K$ and $y_j \in M$ for $j = 1, \dots, m$ such that

$$b_{i1} y_1 + \cdots + b_{im} y_m = x_i$$

for every $i = 1, \dots, n$; that means

$$\mathbf{k}_1 y_1 + \cdots + \mathbf{k}_m y_m = (x_1, \dots, x_n).$$

(**Hint**. Use flatness and the SES in (1))

5. (10 points) Suppose A is a unital commutative ring and $\mathrm{Spec}(A)$ is the set of all the prime ideals of A. For an A-module M and $\mathfrak{p} \in \mathrm{Spec}(A)$, let $M_{\mathfrak{p}}$ be the localization of M at $\mathfrak{p}$. Let

$$\operatorname{supp} M := \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_{\mathfrak{p}} \neq 0\}.$$

Prove that for a finitely generated A-module M,

$$\operatorname{supp} M = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathrm{ann}(M) \subseteq \mathfrak{p}\},$$

where $\mathrm{ann}(M)$ is the annihilator of M.

6. (10 points) Suppose p is a prime. Prove that $x^p - x + 1$ is irreducible in $\mathbb{F}_p[x]$.

7. Suppose F is a field of characteristic zero, $f \in F[x]$ is monic and irreducible, and E is a splitting field of f over F. Let $X := \{\alpha \in E \mid f(\alpha) = 0\}$.

   (a) (3 points) Suppose $m \in \mathbb{N}$ and $\alpha \in X$. Let $g(x) := m_{\alpha^m,F}(x)$ be the minimal polynomial of α^m over F. Prove that $\{\beta^m \mid \beta \in X\}$ is the set of zeros of $g(x)$ in E.

   (b) (4 points) Suppose there exist $\alpha \in E$ and $r \in F$ such that $\alpha, r\alpha \in X$. Prove that

   $$\ell_r : X \to X, \quad \ell_r(\beta) = r\beta$$

   is well-defined. Deduce that r is a root of unity.

   (c) (3 points) Suppose $\alpha, r\alpha \in X$, $r \in F$, and the multiplicative order of r is m. Prove that

   $$m_{\alpha,F}(x) = m_{\alpha^m,F}(x^m); \quad \text{that means} \quad f(x) = g(x^m).$$

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Good Luck!