Name ______________________________ **SID** ______________________________

Qualifying Exam in Numerical Analysis
Spring 2024
Time: 9:00 am – 12:00 noon, Tuesday, May 21, 2024

	Full Scores	**Your Scores**
# 1	25	
# 2	25	
# 3	25	
# 4	25	
# 5	25	
# 6	25	
# 7	25	
# 8	25	
Total	200	

Exam Rules and Instructions

- **This is a three-hour, close-book, and close-note exam. No calculators, computers, tablets, and any other electronic devices are allowed. No cheatsheets are allowed.**
- **There are a total of 8 problems and a total 200 points. There are a total of 13 pages (including this cover page) of the exam.**
- **You must show all the steps for getting the answers. No credit will be given for unsupported answers. All numbers in computational results must be exact, in rational/radical or decimal format. No credit will be given for rounded numbers.**
- **You may use, without proof, any results proved in lectures, but not as homework problems. If you use such a result, please cite it by its name (if it has one) or explain what it is concisely. Please also verify explicitly all the hypotheses in the statement. If the statement you are asked to prove is exactly a result in lecture or text, you still need to provide a proof instead of just citing the result.**

Problem 1. Consider the following matrices and vectors

$$A = \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}, \quad \hat{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 10 \end{bmatrix}, \quad r = b - A\hat{x}.$$

(1) Find the matrix 1-norm $\|A\|_1$.

(2) If $\delta A \in \mathbb{R}^{2\times 2}$ is a perturbation matrix such that $(A + \delta A)\hat{x} = b$, show that

$$\|\delta A\|_1 \geq \frac{\|r\|_1}{\|\hat{x}\|_1}.$$

(3) Use the above results to find the perturbation matrix δA such that $(A + \delta A)\hat{x} = b$ and $\|\delta A\|_1$ is the smallest. Give sufficient reasons to justify your answer.

Problem 2. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix satisfying $\det(A(1\colon k, 1\colon k)) > 0$, for all $1 \le k \le n$.

(1) Prove existence and uniqueness of the factorization $A = LDL^T$, for some unit triangular L, and diagonal matrix D with positive diagonal elements.

(2) Conclude that A is symmetric, positive definite.

Problem 3. Let $n \geq 3$ be a positive integer. Suppose the sequence of vectors $x^{(0)}, x^{(1)}, \ldots, x^{(k)}, \ldots$ in $\mathbb{R}^n$ is generated by the iterative formula

$$x^{(k+1)} = Bx^{(k)} + c$$

for some $B \in \mathbb{R}^{n\times n}$ and $c \in \mathbb{R}^n$. Suppose B has two eigenvalues λ_1, λ_2 such that $|\lambda_1| > 1 > |\lambda_2|$ and 1 is not an eigenvalue of B. Show that there exists an initial vector $x^{(0)}$ such that the sequence $\{x^{(k)}\}_{k=0}^{\infty}$ converges and there exists an initial vector $x^{(0)}$ such that the sequence $\{x^{(k)}\}_{k=0}^{\infty}$ diverges. Give sufficient reasons to justify your answer.

Problem 4. For an integer $n \geq 0$, $\mathcal{P}_n$ denotes the set of all (real) polynomials of degree $\leq n$.

(1) Let $x_0 = -1,\ x_1 = 0,\ x_2 = 1,\ f_0 = -1,\ f_1 = 0,\ f_2 = 1$. Denote $\hat{\mathcal{P}}_2 = \{q \in \mathcal{P}_2 : q(1) = 0\}$. Find $p \in \hat{\mathcal{P}}_2$ such that

$$\sum_{i=0}^{2} [p(x_i) - f_i]^2 \leq \sum_{i=0}^{2} [q(x_i) - f_i]^2 \qquad \forall q \in \hat{\mathcal{P}}_2.$$

(2) Let m, n be two integers such that $m \geq n \geq 1$ and $x_0 < x_1 < \cdots < x_m$.

(i) Show that $\langle p, q \rangle = \sum_{i=0}^{m} p(x_i)q(x_i)$ defines an inner product of $\mathcal{P}_n$.

(ii) Let $Q_0, \ldots, Q_n \in \mathcal{P}_n$ be an orthonormal basis of $\mathcal{P}_n$ with respect to this inner product. Let $f_0, \ldots, f_m \in \mathbb{R}$ and define $p = \sum_{k=0}^{n} \alpha_k Q_k \in \mathcal{P}_n$, where $\alpha_k = \sum_{i=0}^{m} f_i Q_k(x_i)$. Prove that

$$\sum_{i=0}^{m} [p(x_i) - f_i]^2 \leq \sum_{i=0}^{m} [q(x_i) - f_i]^2 \qquad \forall q \in \mathcal{P}_n.$$

Problem 5. Let $a, b \in \mathbb{R}$ with $a < b$ and $n \geq 1$ be an integer. Let $x_0, \ldots, x_n$ be $n+1$ distinct points in $[a, b]$. Let $f \in C([a, b])$ and $\varepsilon > 0$. Prove there exists a polynomial q such that

$$\|f - q\|_{C([a,b])} < \varepsilon \quad \text{and} \quad q(x_i) = f(x_i) \quad (i = 0, \ldots, n).$$

Problem 6. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. Consider the numerical quadrature

$$\int_\alpha^\beta g(t)\,dt \approx \frac{1}{2}(\beta-\alpha)\,[g(\alpha)+g(\beta)] - \frac{1}{12}(\beta-\alpha)^2\,[g'(\beta)-g'(\alpha)] \qquad \forall g \in C^1([\alpha,\beta]).$$

Denote the left-hand and right-hand sides of this formula by $I(g)$ and $\hat{I}(g)$, respectively. It is known that the error of the quadrature is

$$I(g) - \hat{I}(g) = K(\beta-\alpha)^5 g^{(4)}(\xi), \qquad \forall g \in C^{(4)}([\alpha,\beta]),$$

where $K \in \mathbb{R}$ is independent of α, β and g, and $\xi \in [\alpha,\beta]$ depends on g.

(1) Find the degree of precision of this quadrature and the value of K.

(2) Let $a < b$, $N \geq 2$ an integer, $h = (b-a)/N$, and $x_k = a + kh$ $(k = 0, 1, \dots, N)$. Let $f \in C^4([a,b])$. Derive the corresponding composite quadrature and its error formula for $\int_a^b f(x)\,dx$.

Problem 7.

(1) Consider $y \in C^\infty([t_0, T])$, for $t_0 < T$ and consider $t \in [t_0, T)$. Determine a choice of $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$ such that the approximation

$$\frac{\alpha_2 y(t+2h) + \alpha_1 y(t+h) + \alpha_0 y(t)}{h} \approx y'(t)$$

is second-order accurate, as $h \to 0^+$.

(2) Now consider the initial value problem consisting of ODE

$$y' = f(t, y)$$

for $t \in [t_0, T]$, and initial value $y(t_0) = y_0$. Suppose $f(t, y)$ is continuous, and Lipschitz continuous in y, in $\{(t, y) \mid t \in [t_0, T]\}$, and further suppose $y \in C^\infty([t_0, T])$.
Utilizing the derivative approximation of the previous part, write down the difference equation of a linear multistep method for solving the initial value problem that has second-order local truncation error, then determine whether or not the method will be convergent, when given exact starting values.

Problem 8. Consider the initial value problem consisting of ODE

$$y' = f(t, y)$$

for $t \in [t_0, T]$, and initial value $y(t_0) = y_0$. Suppose $f(t, y)$ is continuous, and Lipschitz continuous in y, in $\{(t, y) \mid t \in [t_0, T]\}$.

Also consider explicit 3-stage Runge-Kutta methods, using starting value y_0 at t_0, and consider their Butcher tableaus:

$$\begin{array}{c|ccc} \alpha_1 & 0 & 0 & 0 \\ \alpha_2 & \beta_{21} & 0 & 0 \\ \alpha_3 & \beta_{31} & \beta_{32} & 0 \\ \hline & c_1 & c_2 & c_3 \end{array}$$

(1) Find necessary and sufficient conditions on the Butcher tableau so that a 3-stage explicit Runge-Kutta method will be consistent with the initial value problem.

(2) Prove an explicit 3-stage Runge-Kutta method that is consistent with the initial value problem cannot be A-stable. Hint: note polynomial forms.