Name: ______________________________

PID: ______________________________

Question	Points	Score
1	8	
2	8	
3	8	
4	8	
5	8	
6	8	
7	8	
Total:	56	

1. Write your name on the front page of your exam.
2. Read each question carefully, and answer each question completely.
3. Write your solutions clearly in the exam sheet.
4. Show all of your work; no credit will be given for unsupported answers.
5. You may use the result of one part of the problem in the proof of a later part, even you do not complete the earlier part.

1. (8 points) Suppose p and q are two distinct primes and G is a group of order p^2q. Prove that G is solvable.

2. Suppose G is a finite group. Let $\mathrm{Syl}_p(G)$ be the set of Sylow p-subgroups of G and s_p the number of elements in $\mathrm{Syl}_p(G)$.

   (a) (2 points) Suppose $P, Q \in \mathrm{Syl}_p(G)$ are distinct. Prove that

   $$P \cap N_G(Q) = P \cap Q.$$

   (b) (2 points) Suppose $P \in \mathrm{Syl}_p(G)$ and consider the action of P on $\mathrm{Syl}_p(G)$ by conjugation. Prove that the P-orbit of $Q \in \mathrm{Syl}_p(G)$ has $[P : P \cap Q]$ many elements.

   (c) (4 points) Suppose $p^e \mid (s_p - 1)$ and $p^{e+1} \nmid (s_p - 1)$. Prove that there are distinct $P, Q \in \mathrm{Syl}_p(G)$ such that

   $$[P : P \cap Q] \leq p^e.$$

3. Suppose A is a unital commutative ring.

   (a) (3 points) Suppose $I \trianglelefteq A$ (that means I is an ideal of A) and $a \in A$. Suppose $(I : \langle a\rangle) := \{r \in A \mid ra \in I\}$ and $I + \langle a\rangle$ are finitely generated ideals. Prove that I is finitely generated.

   (b) (2 points) Let $\Sigma := \{I \trianglelefteq A \mid I \text{ is not finitely generated}\}$. Prove that, if Σ is not empty, then with respect to the inclusion ordering, Σ has a maximal element.

   (c) (3 points) Suppose I is a maximal element of Σ. Prove that I is a prime ideal.

4. Suppose A is a unital commutative ring and $I, J \trianglelefteq A$. Suppose A/I is a flat A-module.

   (a) (2 points) Prove that I is a flat A-module. (State carefully the general statement that you are using.)

   (b) (2 points) Prove that there is an A-module isomorphism $\iota : I \otimes_A J \to IJ$ such that $\iota(a \otimes b) = ab$ for every $a \in I$ and $b \in J$.

   (c) (4 points) Prove that $IJ = I \cap J$.

5. Suppose A is a Noetherian local ring and its only maximal ideal is $\mathfrak{m}$.

   (a) (1 point) Carefully state Nakayama's lemma for A-modules.

   (b) (3 points) For every finitely generated A-module M, prove that

$$d(M) = \dim_{A/\mathfrak{m}}(M/\mathfrak{m}M),$$

   where $d(M)$ is the minimum number of elements needed to generate M.

   (c) (4 points) Prove that every finitely generated projective A-module P is free.

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6. Suppose F is a field of characteristic zero and f is an irreducible polynomial of degree n in $F[x]$. Let E be a splitting field of f over F. Let $\{\alpha_1, \ldots, \alpha_n\}$ be the set of zeros of f in E.

   (a) (5 points) Prove that if $\mathrm{Gal}(E/F)$ is abelian, then $E = F[\alpha_i]$ for every i.

   (b) (3 points) Prove that if $n = p$ is prime, then $E = F[\alpha_1]$ implies that $\mathrm{Gal}(E/F) \simeq \mathbb{Z}/p\mathbb{Z}$.

7. (8 points) Let K be a field with 729 elements. Let $\mathbb{F}_3$ be the prime field of K and set $G = \mathrm{Gal}(K/\mathbb{F}_3)$. Consider the action of G on K given by $\sigma \cdot u = \sigma(u)$. Describe the orbits of this action, calculate their sizes, and calculate how many orbits are there of each size.

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Good Luck!