Name: ______________________________ S.I.D.: ______________________________

Qualifier Exam in Applied Algebra

May 23, 2024

	Full	Real
# 1	10	
# 2	10	
# 3	10	
# 4	10	
# 5	10	
# 6	10	
# 7	10	
# 8	10	
# 9	10	
# 10	10	
Total	100	

Notes: 1) For computational questions, no credit will be given for unsupported answers obtained directly from a calculator. 2) For proof questions, no credit will be given for no reasons or wrong reasons.

1. (10 points) Let $A \in M_n(\mathbb{C}) = \mathbb{C}^{n\times n}$ and let $x \in \mathbb{C}^n, x \neq 0$. Prove if $S = (\text{span}\{x\})^\perp \subseteq \mathbb{C}^n$ is an invariant subspace of A, meaning $AS \subseteq S$, then x is an eigenvector of A^H.

   Note: $A^H = \overline{A^T}$.

2. (10 points) Let $A = (a_{ij}) \in M_n(\mathbb{C}) = \mathbb{C}^{n\times n}$ and define the sets

$$D_i = \left\{ \beta \in \mathbb{C} \;\middle|\; |\beta - a_{ii}| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \right\} \subseteq \mathbb{C},$$

for $1 \leq i \leq n$. Given any λ an eigenvalue of A, prove

$$\lambda \in \bigcup_{i=1}^{n} D_i.$$

3. (10 points) Let $A \in M_n(\mathbb{C}) = \mathbb{C}^{n \times n}$. Prove A is Hermitian if and only if $x^H A x \in \mathbb{R}$, for all $x \in \mathbb{C}^n$.
   Note: $x^H = \overline{x^T}$.

4. (10 points) Given $A \in M_n(\mathbb{C}) = \mathbb{C}^{n\times n}$ an upper triangular matrix and $\epsilon \in \mathbb{R}, \epsilon > 0$, prove there exists $\eta \in \mathbb{R}, \eta > 0$ such that the diagonal matrix $D = (d_{ij}) \in M_n(\mathbb{R}) = \mathbb{R}^{n\times n}$ with entries

$$d_{jj} = \eta^{j-1},$$

for $1 \le j \le n$, satisfies

$$\|A\| \le \rho(A) + \epsilon,$$

where the matrix norm $\|\cdot\|$ is defined by:

$$\|B\| = \|D^{-1}BD\|_1,$$

for all $B \in M_n(\mathbb{C}) = \mathbb{C}^{n\times n}$.

5. (10 points) Give a complete statement and proof of Schur's Lemma, in the category of complex finite-dimensional representations of finite groups.

6. (10 points) Let (V, φ) and (W, ψ) be finite-dimensional unitary representations of a finite group G. Let $\mathcal{L}(\mathrm{V}, \mathrm{W}) = \{\text{linear } T \colon \mathrm{V} \to \mathrm{W}\}$ equipped with the scalar product $\langle S, T \rangle = \mathrm{Tr}\ S^*T$. For $g \in \mathrm{G}$ and $T \in \mathcal{L}(\mathrm{V}, \mathrm{W})$, put $\omega(g)T = \psi(g)T\varphi(g^{-1})$. Show that $(\mathcal{L}(\mathrm{V}, \mathrm{W}), \omega)$ is a unitary representation of G. Compute the character of $(\mathcal{L}(\mathrm{V}, \mathrm{W}), \omega)$ in terms of the characters of (V, φ) and (W, ψ), showing your calculations.

7. (10 points) Let (V, φ) be a finite-dimensional unitary representation of a finite group G. State the definition of the space V^{G} of G-invariant vectors in V, and prove that $P = \frac{1}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} \varphi(g)$ is the orthogonal projection of V onto V^{G}.

8. (10 points) State the definition of the standard representation of the symmetric group and compute its character in terms of the enumeration of fixed points in permutations. Using character theory or otherwise, prove that the standard representation is irreducible.

9. (10 points) Let $I \subseteq \mathbb{C}[x,y]$ be an ideal such that $\mathbf{V}(I) = \{(1,3)\}$. Prove that $\mathbb{C}[x,y]/I$ is a finite-dimensional vector space. Is this true if $\mathbb{C}[x,y]$ is replaced by $\mathbb{R}[x,y]$?

10. (10 points) Let n be a positive integer and let $G \subseteq GL_n(\mathbb{C})$ be the subgroup

$$G = \left\{ \begin{pmatrix} \epsilon_1 & 0 & \cdots & 0 \\ 0 & \epsilon_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \epsilon_n \end{pmatrix} : \epsilon_1, \epsilon_2, \ldots, \epsilon_n = \pm 1 \right\}$$

of diagonal matrices whose diagonal entries are ± 1. Find the Hilbert series of the invariant ring $\mathbb{C}[x_1, x_2, \ldots, x_n]^G$.