Instructions: This is a closed-book examination. You have 180 minutes to complete the test. You may use without proof results proved in Conway up to and including Chapter XI. When using a result from the text, be sure to explicitly verify all hypotheses in it. Present your solutions clearly, with appropriate detail. If using a homework problem, please make sure you reprove it.

Notation and terminology: The unit disk \( \{ |z| < 1 \} \) is denoted by \( \mathbb{D} \). A region is an open and connected subset of \( \mathbb{C} \).

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Let $f$ be a holomorphic function on a region $G$ such that $|f|^2 + |f|$ is harmonic on $G$. Prove $f$ is constant.
2. [10] Let $f : \mathbb{C} \to \mathbb{C}$ be an entire nowhere zero function. Define $U = \{ z : |f(z)| < 1 \}$. If $U \neq \emptyset$, show that the connected components of $U$ are unbounded.
3. [10] Let \( f : \mathbb{D} \to \mathbb{C} \) be holomorphic. Assume \( \text{Re} \, f(z) > 0 \) for all \( z \in \mathbb{D} \). Show that

\[
|f'(0)| \leq 2 \text{Re} \, f(0).
\]
Let $\phi$ be a positive harmonic function on a simply connected region $G$. Prove that there are two harmonic functions $u, v$ on $G$ such that $\phi = e^u \sin v$. 
5. [10] Show that there exist polynomials $p_n$ such that

(i) $p_n(0) = 1$, $p_n'(0) = 0$,

(ii) $p_n(z) \to 0$ as $n \to \infty$ for all fixed $z \in \mathbb{C} \setminus \{0\}$. 
6. [4, 6] Let \( U \subset \mathbb{C} \) be a bounded connected open set containing 0, and \( f : U \to U \) a holomorphic function which satisfies \( f(0) = 0 \) and \( |f'(0)| < 1 \). Write
\[
f^{(n)} = f \circ f \circ \ldots \circ f. \]

(i) Show that there is a neighborhood \( V \) of 0 such that the sequence \( f^{(n)} \) converges to 0 locally uniformly on \( V \).

\textit{Hint:} \( |f(z)| \leq M|z| \) for a constant \( M < 1 \), for \( |z| \) small.

(ii) Show that the sequence \( f^{(n)} \) converges locally uniformly to 0 on \( U \).
7. [2, 4, 4] Let $U \subset \mathbb{C}$ be an open set. $f : U \setminus \{a\} \to \mathbb{C}$ be a holomorphic function with an isolated singularity at $a \in U$.

Let $P$ be a non-constant polynomial. Let $g : U \setminus \{a\} \to \mathbb{C}$ be given by

$$g(z) = P(f(z)).$$

Show that:

(i) If $f$ has a removable singularity at $a$, then $g$ has a removable singularity at $a$.

(ii) If $f$ has a pole at $a$, then $g$ has a pole at $a$.

(iii) If $f$ has an essential singularity at $a$, then $g$ has an essential singularity at $a$. 