Name: ____________________

PID: ____________________

Question	Points	Score
1	7	
2	10	
3	15	
4	10	
5	8	
6	7	
7	5	
8	8	
Total:	70	

1. Write your Name on the front page of your exam.
2. Read each question carefully, and answer each question completely.
3. Write your solutions clearly in the exam sheet.
4. Show all of your work; no credit will be given for unsupported answers.
5. You may use the result of one part of the problem in the proof of a later part, even you do not complete the earlier part.

1. Suppose G is a nilpotent group.
   (a) (5 points) Prove that for every proper subgroup H of G, $H \neq N_G(H)$.

   (b) (2 points) Prove that every maximal subgroup of G is normal.

2. Suppose G is a finite group, p is prime, m is an integer, $\gcd(p, m) = 1$, and $|G| = pm$. Suppose P is a Sylow p-subgroup and $N_G(P) = P$.

   (a) (4 points) Prove that if G has a subgroup H of order m, then

   $$H = \{x \in G \mid o(x) \neq p\}.$$

   Deduce that in this case, H is a characteristic subgroup.

   (b) (1 point) Suppose G is solvable. Prove that G has a normal subgroup N such that $G/N \simeq \mathbb{Z}/\ell\mathbb{Z}$ for some prime ℓ.

   (c) (5 points) Suppose G is solvable. Prove that G has a normal subgroup of order m. (**Hint.** Use induction on $|G|$, the subgroup N from the previous part, and a Sylow ℓ-subgroup of N if needed.)

3. Suppose $A = \mathbb{Z}[\sqrt{-5}]$.

   (a) (5 points) Prove that 3 is irreducible in A, but it is not prime.

   (b) (5 points) Prove that $\mathfrak{a} := \langle 1 + \sqrt{-5}, 3 \rangle$ is **not** a free A-module.

(c) (5 points) Prove that $\mathfrak{a}$ is a projective A-module.

4. Suppose A is a unital commutative ring.

   (a) (5 points) Suppose P_1 and P_2 are projective A-modules. Prove that

   $$P_1 \otimes_A P_2$$

   is a projective A-module.

   (b) (5 points) Suppose M_1 and M_2 are flat A-modules. Prove that $M_1 \otimes_A M_2$ is a flat A-module.

5. Suppose p is a prime number and the minimal polynomial of $a \in \mathrm{M}_p(F)$ is $t^p - 1$.

   (a) (4 points) Find the Jordan form of a if $F = \mathbb{C}$. Justify your answer.

   (b) (4 points) Find the Jordan form of a if $F = \mathbb{F}_p$. Justify your answer.

6. Suppose A is a finitely generated $\mathbb{Q}$-algebra, and $\phi : A \to A$ is a surjective ring homomorphism.

   (a) (2 points) Prove that A is a Noetherian ring.

   (b) (5 points) Prove that ϕ is an isomorphism. (**Hint.** Consider $\{\ker \phi^n\}_{n=1}^{\infty}$.)

7. (5 points) Suppose F is a field, $f \in F[x]$ is irreducible, and E is a splitting field of f over F. Suppose there exists $\alpha \in E$ such that $f(\alpha) = f(\alpha + 1) = 0$. Prove that the characteristic of F is positive.

8. Suppose p is a prime and $p \nmid n$. Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$. Suppose $\zeta \in \overline{\mathbb{F}}_p^{\times}$ is an element of (multiplicative) order n.

   (a) (2 points) Prove that $x^n - 1 = \prod_{i=0}^{n-1}(x - \zeta^i)$ in $\overline{\mathbb{F}}_p$.

   (b) (3 points) Let $E_{n,p} \subseteq \overline{\mathbb{F}}_p$ be a splitting field of $x^n - 1$ over $\mathbb{F}_p$. Prove that $E_{n,p}/\mathbb{F}_p$ is a Galois extension and $\mathrm{Gal}(E_{n,p}/\mathbb{F}_p)$ can be embedded into the group of automorphisms $\mathrm{Aut}(\langle\zeta\rangle)$ of the cyclic group $\langle\zeta\rangle$.

   (c) (3 points) Prove that $\mathrm{Gal}(E_{n,p}/\mathbb{F}_p)$ is isomorphic to the cyclic subgroup of $(\mathbb{Z}/n\mathbb{Z})^{\times}$ which is generated by $p + n\mathbb{Z}$.

Good Luck!