Name: ____________________

PID: ____________________

Question	Points	Score
1	10	
2	10	
3	5	
4	10	
5	7	
6	8	
7	10	
8	10	
Total:	70	

1. Write your Name and PID, on the front page of your exam.
2. Read each question carefully, and answer each question completely.
3. Write your solutions clearly in the exam sheet.
4. Show all of your work; no credit will be given for unsupported answers.
5. You may use the result of one part of the problem in the proof of a later part, even if you do not complete the earlier part.
6. You may use major theorems *proved* in class, but not if the whole point of the problem is to reproduce the proof of a theorem proved in class or the textbook. Similarly, quote the result of a homework exercise only if the result of the exercise is a fundamental fact and reproducing the result of the exercise is not the main point of the problem.

1. Suppose G is a group of order 55.

   (a) (5 points) Classify possible structures of the group G.

   (b) (2 points) Is G always solvable? Justify your answer.

   (c) (3 points) Is G always nilpotent? Justify your answer.

2. Suppose G is a finite group of order $p^k m$ where p is prime and $p \nmid m$. Let $\mathrm{Syl}_p(G)$ be the set of all the Sylow p-subgroups of G. Suppose $P_1 \cap P_2 = \{1\}$ for every two distinct Sylow p-subgroups P_1 and P_2.

   (a) (5 points) Prove that $N_G(P_1) \cap P_2 = \{1\}$, where $P_1, P_2 \in \mathrm{Syl}_p(G)$ are distinct. (Derive this based on Sylow's theorems.)

   (b) (3 points) Suppose $P_0 \in \mathrm{Syl}_p(G)$ and consider its action on $\mathrm{Syl}_p(G)$ by conjugation. For $P \in \mathrm{Syl}_p(G)$, let $\mathcal{O}_P$ be the P_0-orbit of P. Prove that if $P \neq P_0$, then $|\mathcal{O}_P| = p^k$.

   (c) (2 points) Prove that $|G| \equiv 1 \pmod{p^k}$.

3. (5 points) Suppose A is a unital commutative ring, M and N are A-modules, and $f : M \to N$ is an A-module homomorphism. For every maximal ideal $\mathfrak{m}$ of A, let $M_{\mathfrak{m}}$ and $N_{\mathfrak{m}}$ be the localizations of M and N at $\mathfrak{m}$, respectively. Recall that
$$f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}, \quad f_{\mathfrak{m}}\left(\frac{x}{s}\right) := \frac{f(x)}{s}$$
is an $A_{\mathfrak{m}}$-module homomorphism. Prove that if $f_{\mathfrak{m}}$ is injective, for all maximal ideals $\mathfrak{m}$, then f is injective.

4. In this question, briefly justify your answers by stating the main relevant results.

   (a) (5 points) Suppose D is a PID. Prove that a finitely generated D-module is flat if and only if it is free.

   (b) (5 points) Prove that every projective A-module is flat, where A is a unital commutative ring.

5. Suppose D is a Noetherian integral domain and F is its field of fractions. Suppose $\mathfrak{a}$ is an ideal of D. Recall that every D-module homomorphism $\phi : \mathfrak{a} \to D$ is of the form $\phi(x) = cx$ for some $c \in F$.

   (a) (2 points) Briefly argue why there exists a short exact sequence of D-modules of the following form

   $$0 \to K \to D^n \to \mathfrak{a} \to 0.$$

   (b) (5 points) Prove that if $\mathfrak{a}$ is a projective D-module, then there exists a finitely generated D-submodule $\mathfrak{c}$ of F such that $\mathfrak{c}\mathfrak{a} = D$ where

   $$\mathfrak{c}\mathfrak{a} := \left\{ \sum_{i=1}^{m} c_i x_i \mid c_i \in \mathfrak{c}, x_i \in \mathfrak{a} \right\}.$$

6. (8 points) Suppose E/F is a separable extension and $[E : F] = n$. Suppose $\overline{E}$ is an algebraic closure of E. Prove that

$$E \otimes_F \overline{E} \simeq \underbrace{\overline{E} \oplus \cdots \oplus \overline{E}}_{n \text{ times}}$$

as rings.

7. Suppose p is an odd prime and $\zeta_p := e^{2\pi i/p} = \cos(2\pi/p) + i\sin(2\pi/p)$.

(a) (2 points) Prove that $[\mathbb{Q}[\zeta_p] : \mathbb{Q}[\cos(\frac{2\pi}{p})]] = 2$.

(b) (6 points) Prove that $\mathbb{Q}[\cos(\frac{2\pi}{p})]/\mathbb{Q}$ is a Galois extension and its Galois group is a cyclic group of order $(p-1)/2$.

(c) (2 points) Does there exist an odd prime p such that $\sqrt[3]{2} \in \mathbb{Q}[\cos(\frac{2\pi}{p})]$?

8. Suppose n is a positive integer, p is a prime number, and $a \in \mathbb{F}_p^\times$. Suppose $n|p-1$. Suppose E is a splitting field of $x^n - a$ over $\mathbb{F}_p$.

   (a) (6 points) Prove that $E/\mathbb{F}_p$ is a Galois extension and $\mathrm{Gal}(E/\mathbb{F}_p)$ can be embedded into $\mathbb{F}_p^\times$.

   (b) (4 points) Prove that $\mathrm{Gal}(E/\mathbb{F}_p) \simeq \langle a^{\frac{p-1}{n}} \rangle \subseteq \mathbb{F}_p^\times$.

Good Luck!

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