## Applied Algebra Qualifying Exam: Part B Fall 2023

**Instructions:** Do all problems. All problems are weighted equally. You are not allowed to consult any external resource during this exam. Good luck!

**Problem 1:** Let G be a finite group and let V be a finite-dimensional G-module over  $\mathbb{C}$ . Let  $\chi: G \to \mathbb{C}$  be the character of V and consider the linear operator  $\varphi: V \to V$  given by

$$\varphi(v) := \sum_{g \in G} \overline{\chi(g)} \left( g \cdot v \right)$$

where  $\overline{\chi(g)}$  is the complex conjugate of  $\chi(g)$ . Assume that V is irreducible.

Prove that there exists a complex number  $c \in \mathbb{C}$  such that  $\varphi(v) = cv$  for all  $v \in V$  and find the value of c.

**Problem 2:** Find the character table of the alternating group  $A_4$  of even permutations in  $S_4$ .

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**Problem 4:** Let V be the vector space of polynomials  $f(x_1, x_2, x_3, x_4, x_5)$  in 5 variables  $\{x_1, \ldots, x_5\}$  of homogeneous degree 2 with complex coefficients. The symmetric group  $S_5$  acts on V by variable permutation. Find the decomposition of V into  $S_5$ -irreducibles.